Robust Simulation of Transient PDE Models under Uncertainty

Chenyu Wang, Matthew D. Stuber

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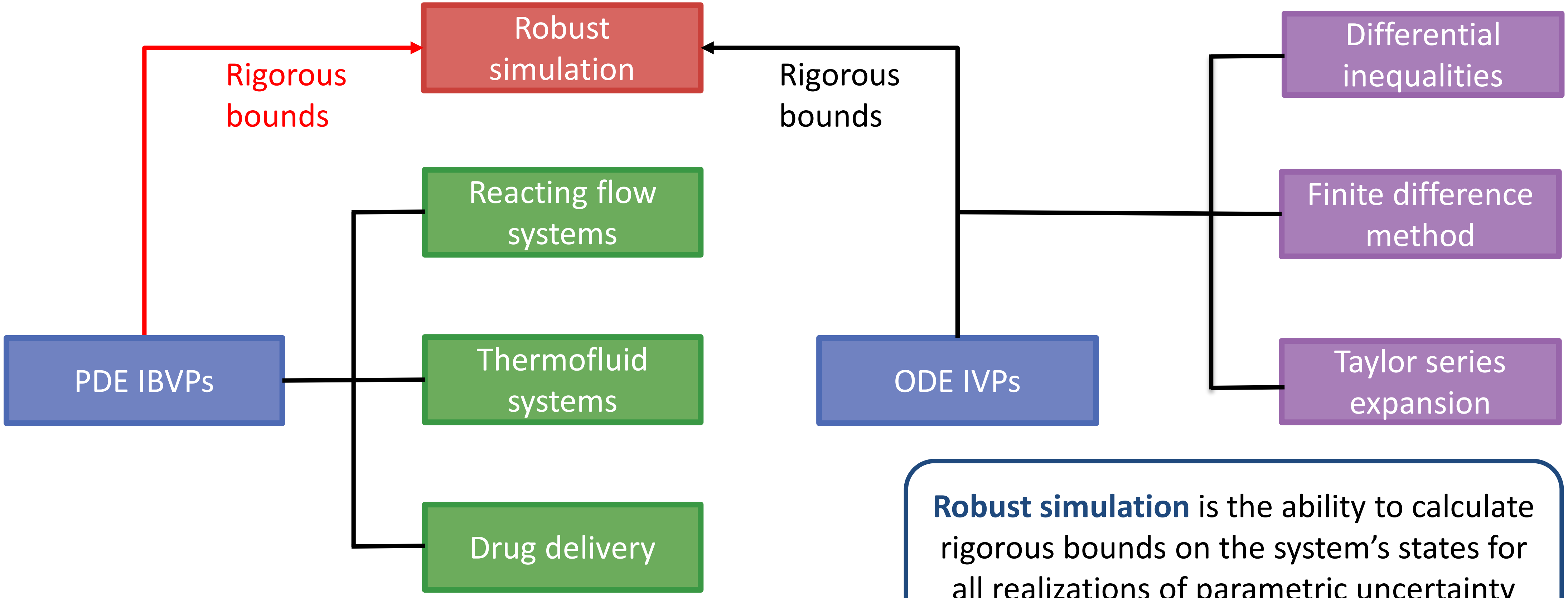
Robust Simulation



Robust simulation is the ability to calculate rigorous bounds on the system's states for all realizations of parametric uncertainty

[1] M. D. Stuber and P. I. Barton, "Robust simulation and design using semi-infinite programs with implicit functions," *International Journal of Reliability and Safety*, vol. 5, no. 3-4, pp. 378-397, 2011.
[2] J. K. Scott and P. I. Barton, "Bounds on the reachable sets of nonlinear control systems," *Automatica*, vol. 49, no. 1, pp. 93-100, 2013.
[3] G. F. Corliss, "Survey of interval algorithms for ordinary differential equations," *Applied Mathematics and Computation*, vol. 31, pp. 112-120, 1989.

Parametric Partial Differential Equations

- Provide valid and efficient bounds for parametric PDE system (IBVP)

$$\partial_t \mathbf{x}(y,\mathbf{p},t) = \mathbf{f}(y,\mathbf{x}(y,\mathbf{p},t),\partial_y \mathbf{x}(y,\mathbf{p},t),\partial_{yy}\mathbf{x}(y,\mathbf{p},t),\mathbf{p},t)$$

$$\mathbf{x}(y(\mathbf{p},t_0),t_0) = \mathbf{x}_0(\mathbf{p}),\ \ \mathbf{p} \in P$$

$$\mathbf{x}(y^L(\mathbf{p},t),t) = \mathbf{x}^L(\mathbf{p}),\ \ \mathbf{p} \in P$$

$$\mathbf{x}(y^U(\mathbf{p},t),t) = \mathbf{x}^U(\mathbf{p}),\ \ \mathbf{p} \in P$$

- Assumptions:
  - $\mathbf{f} : C \times D \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times P \times I \to \mathbb{R}^{n_x}$ is locally Lipschitz continuous;
  - For every $\mathbf{p} \in P$, there exists a unique solution over the time domain I.
- Reachable set: $Re(y,t) \equiv \{\mathbf{x}(y,\mathbf{p},t) : \mathbf{p} \in P\}$.
- Objective: efficiently compute bounds on the reachable set with less conservatism.

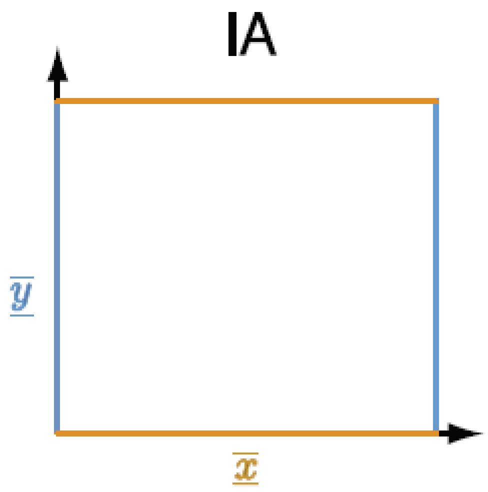
Interval Arithmetic and Affine Arithmetic

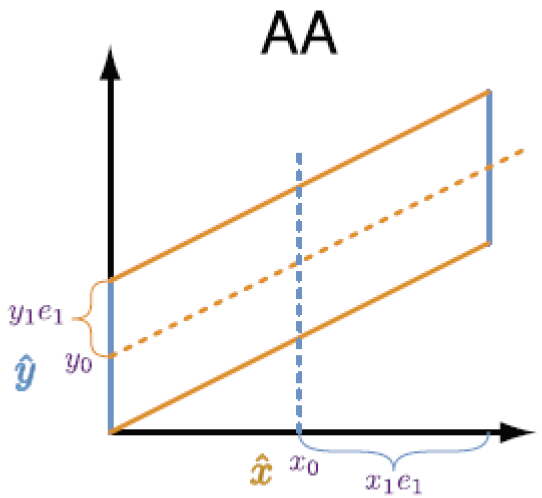
Interval Arithmetic (IA)

Affine Arithmetic (AA)

$$x \in [x^L, x^U] = X \quad \longrightarrow \quad \hat{X} = x_0 + \sum_{i=1}^{n} x_i \varepsilon_i,$$

$$x_0 = (x^L + x^U)/2,$$
$$x_1 = (x^U - x^L)/2,$$
$$x_i = 0, \forall i > 1.$$



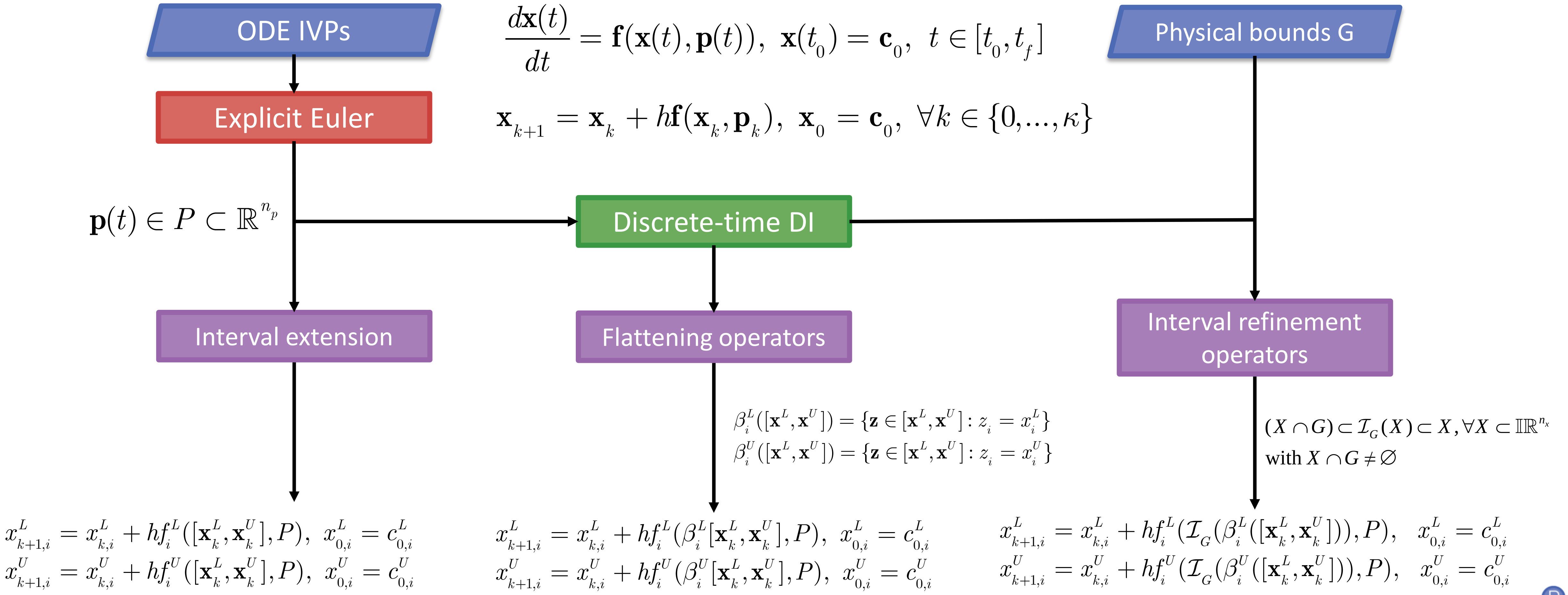


- $F(X) \supseteq \mathbf{f}(X) = \{\mathbf{f}(\mathbf{x}) | \mathbf{x} \in X\}$
- Dependency problems: overestimation

- Keep track of the dependency;
- Enhance the efficacy of interval operations

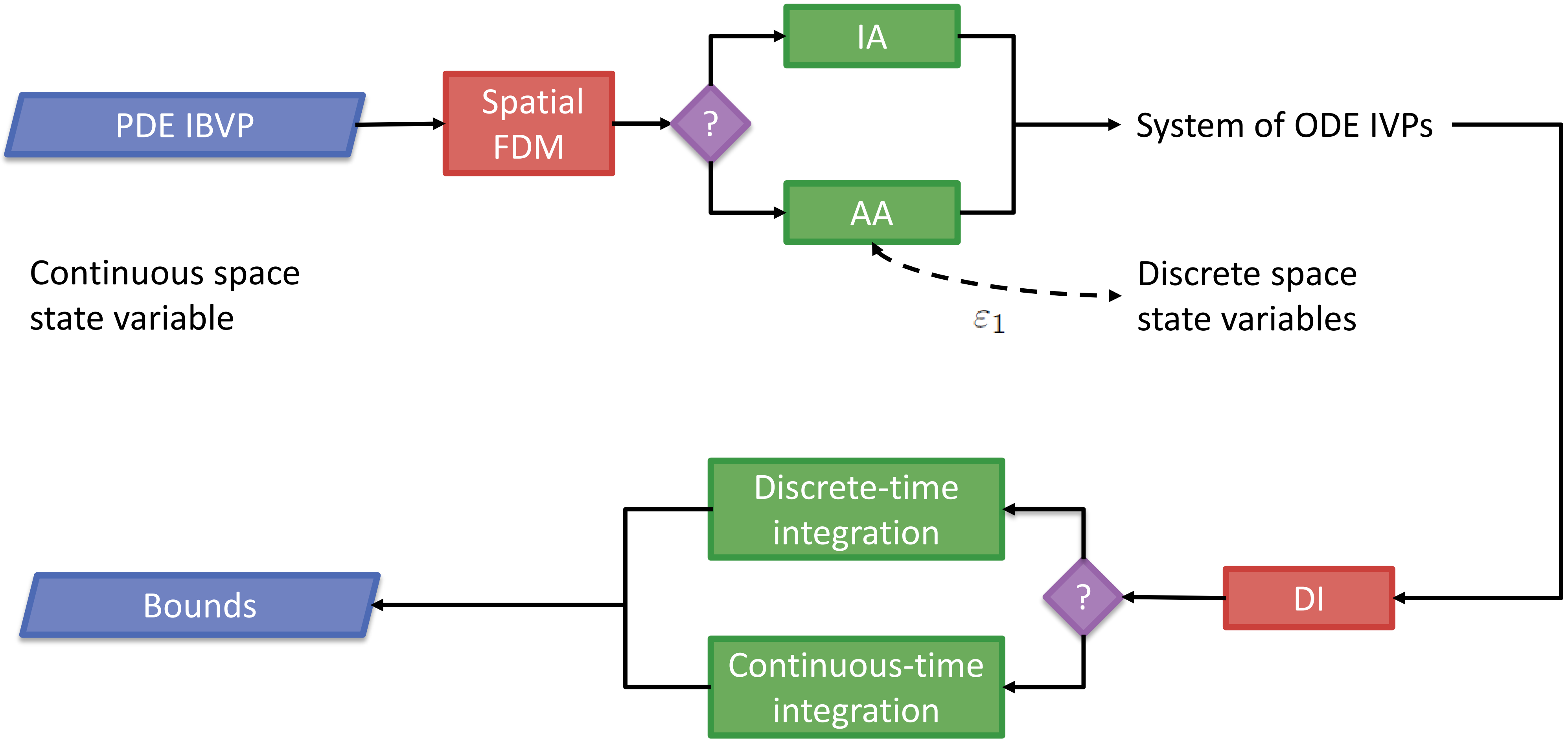
[4] J. Comba, J. Stolfi, “Affine arithmetic and its applications to computer graphics,” *anais do vii sibgrapi*, 9-18 (1993).
[5] J. Stolfi, L. H. de Figueiredo, “An introduction to affine arithmetic,” *Trends in Applied and ComputationalMathematics*, 4 (3) (2003) 297–312.
[6] L. H. De Figueiredo, J. Stolfi, “Affine arithmetic: concepts and applications,” *Numerical Algorithms*, 37 (1-4) (2004) 147–158.

Differential Inequalities



[7] X. Yang and J. K. Scott, “Efficient reachability bounds for discrete-time nonlinear systems by extending the continuous-time theory of differential inequalities," in 2018 Annual American Control Conference (ACC), pp. 6242-6247, IEEE, 2018.

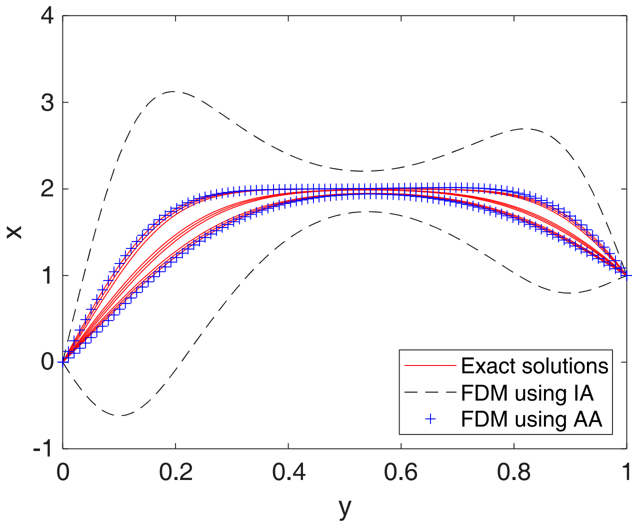
Bounding PDE



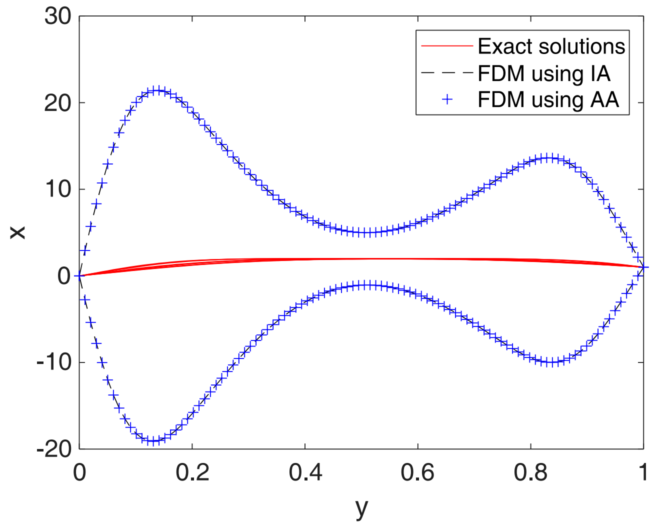
Convection-Diffusion System

$$\frac{\partial x}{\partial t} = p_1 \frac{\partial^2 x}{\partial y^2} - p_2 \frac{\partial x}{\partial y}, \; t \in [0,1], \; y \in [0,1] \qquad p_1 \in [0.1, 0.3] \quad p_2 \in [0.2, 0.6]$$

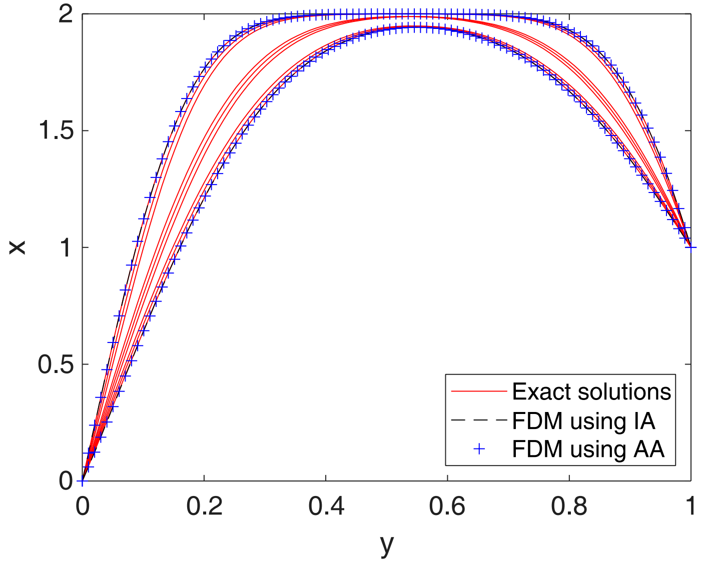
t = 0.5



Centered finite difference



Forward finite difference

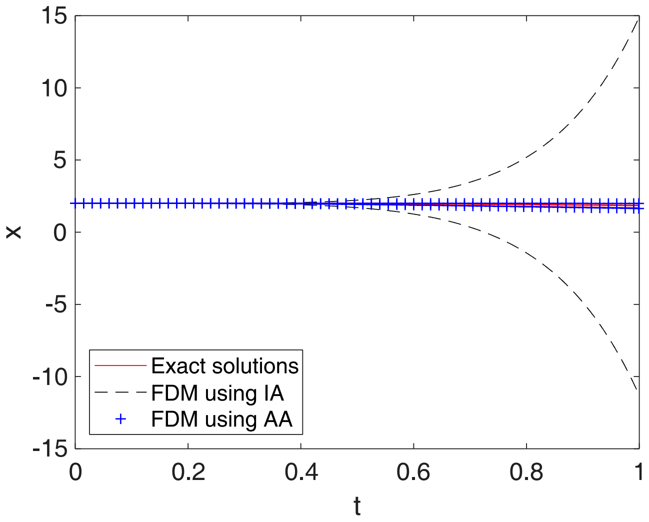


Backward finite difference

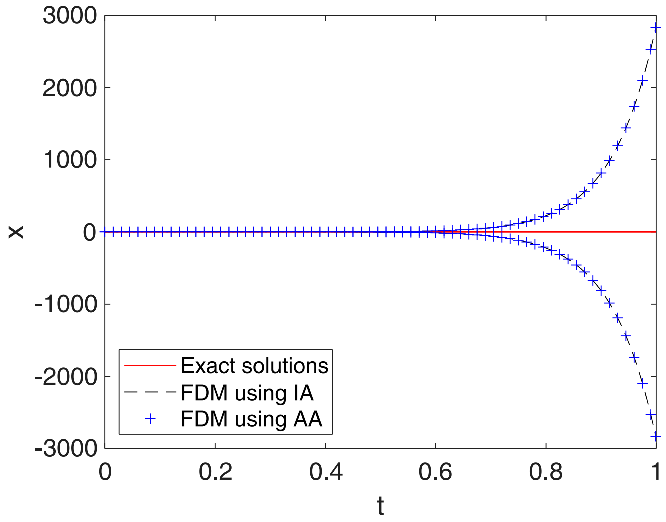
Convection-Diffusion System

$$\frac{\partial x}{\partial t} = p_1 \frac{\partial^2 x}{\partial y^2} - p_2 \frac{\partial x}{\partial y}, \;\; t \in [0,1], \;\; y \in [0,1] \qquad p_1 \in [0.1, 0.3] \quad p_2 \in [0.2, 0.6]$$

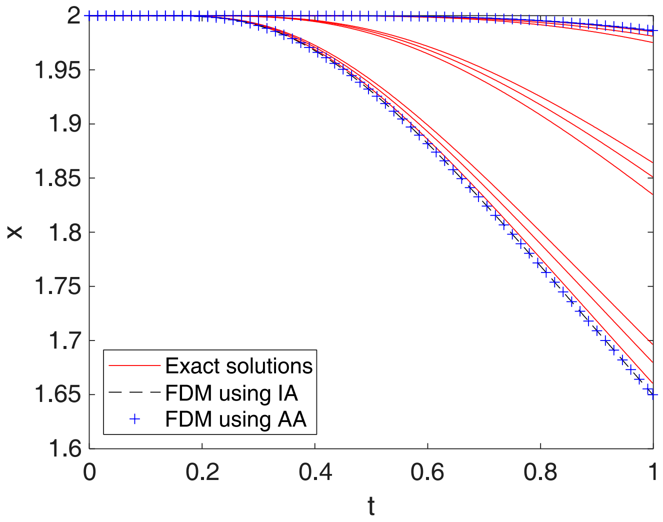
y = 0.5



Centered finite difference



Forward finite difference

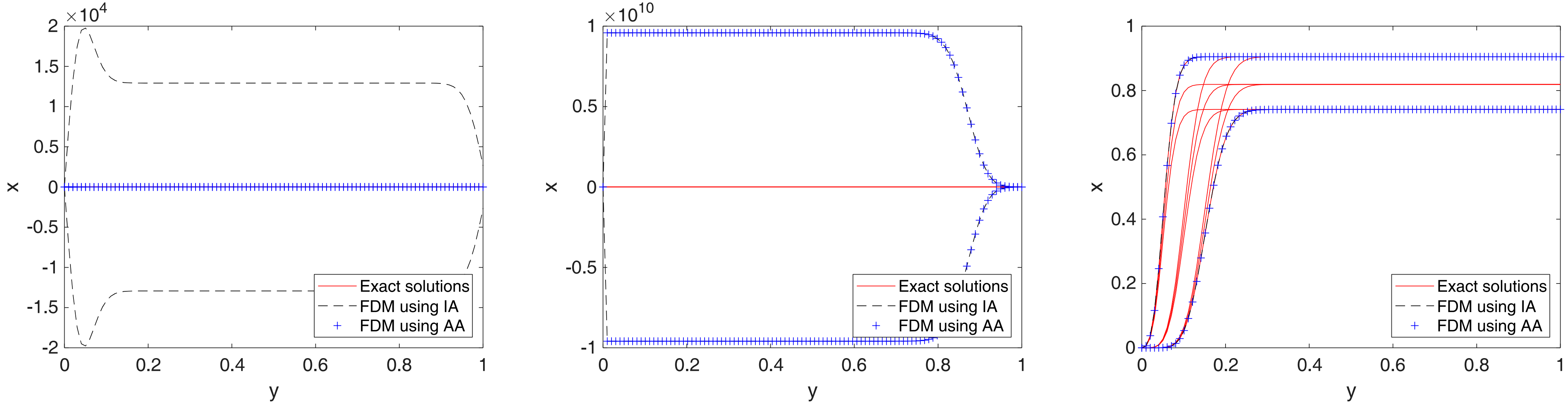


Backward finite difference

Convection-Reaction System

$$\frac{\partial x}{\partial t} = -p_1 \frac{\partial x}{\partial y} - p_2 x, \; t \in [0,1], \; y \in [0,1], \qquad p_1 \in [0.1, 0.3] \quad p_2 \in [0.2, 0.6]$$

t = 0.5

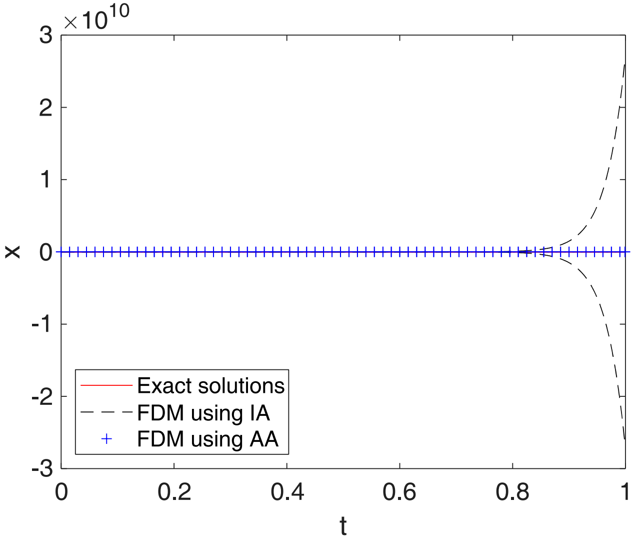


Centered finite difference   Forward finite difference   Backward finite difference

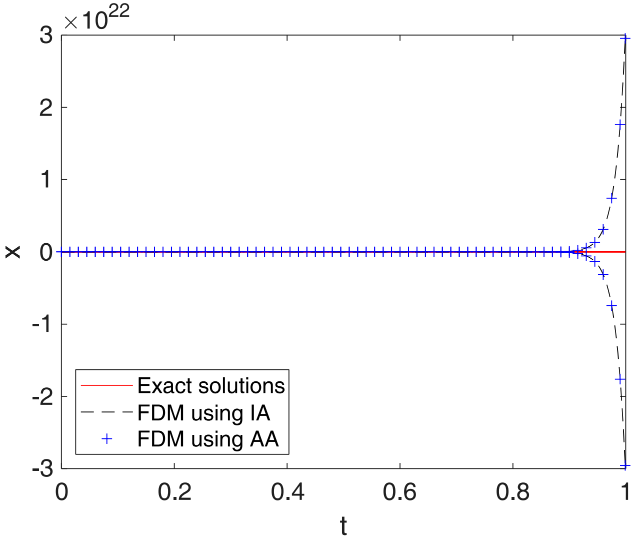
Convection-Reaction System

$$\frac{\partial x}{\partial t} = -p_1 \frac{\partial x}{\partial y} - p_2 x,\ \ t \in [0,1],\ \ y \in [0,1], \qquad p_1 \in [0.1, 0.3] \quad p_2 \in [0.2, 0.6]$$

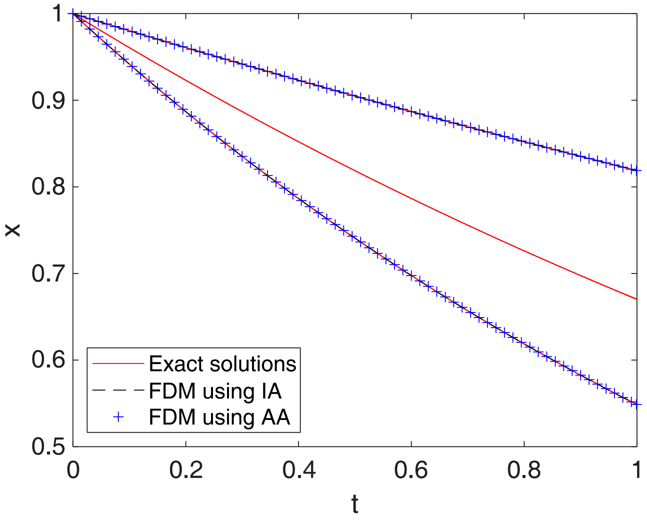
y = 0.5



Centered finite difference



Forward finite difference

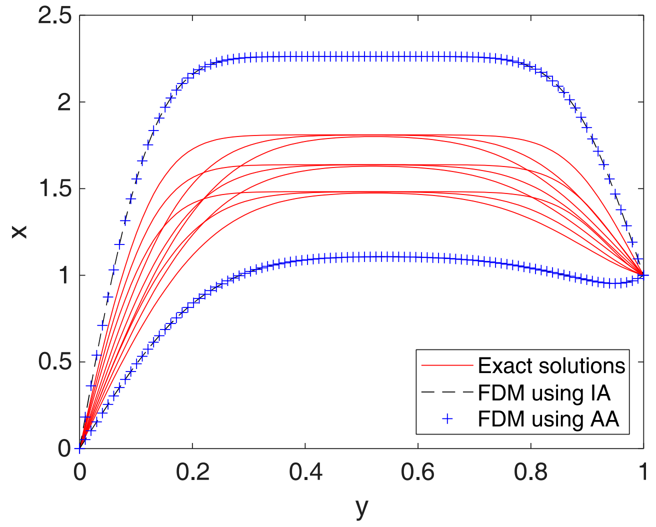


Backward finite difference

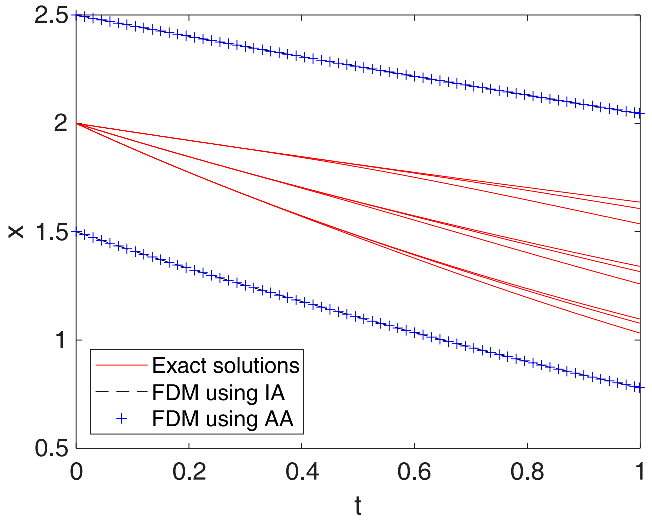
Diffusion-Reaction System

$$\frac{\partial x}{\partial t} = p_1 \frac{\partial^2 x}{\partial y^2} - p_2 x, \;\; t \in [0,1], \;\; y \in [0,1] \qquad p_1 \in [0.01, 0.03] \qquad p_2 \in [0.2, 0.6]$$

t = 0.5



y = 0.5



Coupled IBVPs

$$O + O_2 \xrightarrow{k_1} O_3$$

$$O + O_3 \xrightarrow{k_2} 2O_2$$

$$O_2 \xrightarrow{k_3(t)} 2O$$

$$O_3 \xrightarrow{k_4(t)} O + O_2$$

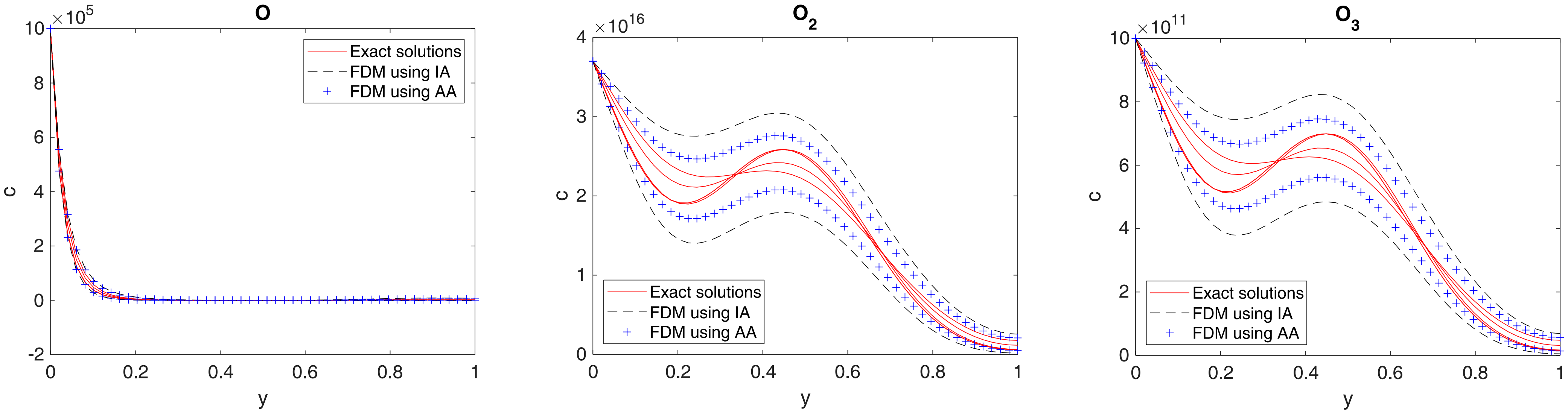
$$\frac{\partial c_O}{\partial t} + u_1 \frac{\partial c_O}{\partial y} = D_1 \frac{\partial^2 c_O}{\partial y^2} + r_{c_O},$$

$$\frac{\partial c_{O_2}}{\partial t} + u_2 \frac{\partial c_{O_2}}{\partial y} = D_2 \frac{\partial^2 c_{O_2}}{\partial y^2} + r_{c_{O_2}},$$

$$\frac{\partial c_{O_3}}{\partial t} + u_3 \frac{\partial c_{O_3}}{\partial y} = D_3 \frac{\partial^2 c_{O_3}}{\partial y^2} + r_{c_{O_3}},$$

$u \in [4\text{e-}3, 6\text{e-}3]$

$D \in [4\text{e-}3, 6\text{e-}3]$



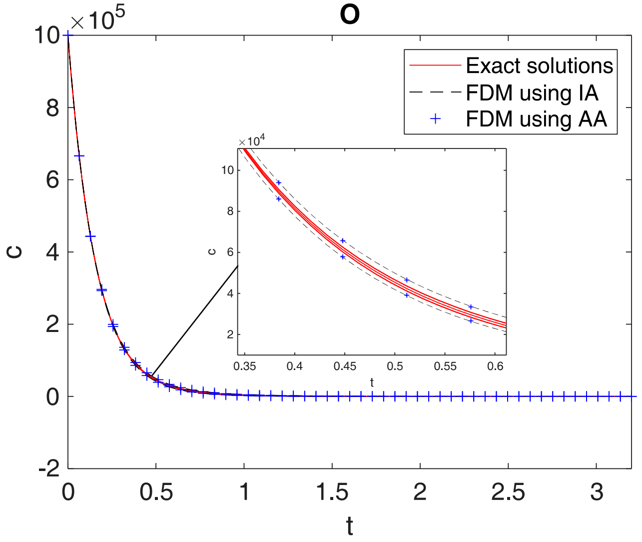
[8] J. Makungu, H. Haario, W. C. Mahera, "A generalized 1-dimensional particle transport method for convection diffusion reaction model," *Afrika Matematika*, 23 (1) (2012) 21–39.

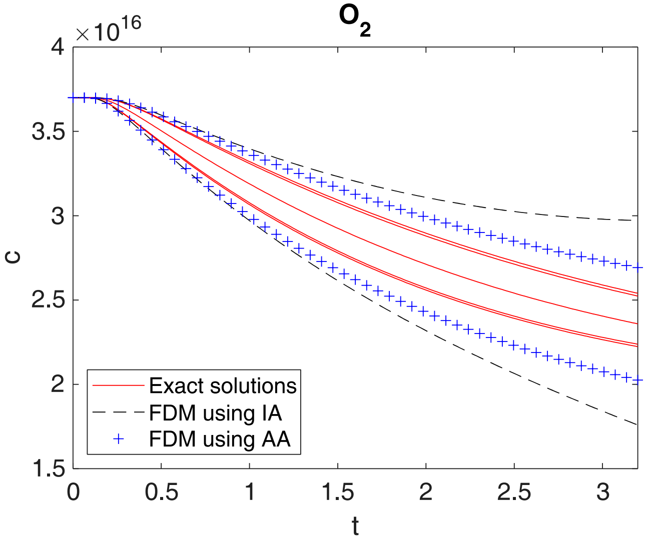
Coupled IBVPs

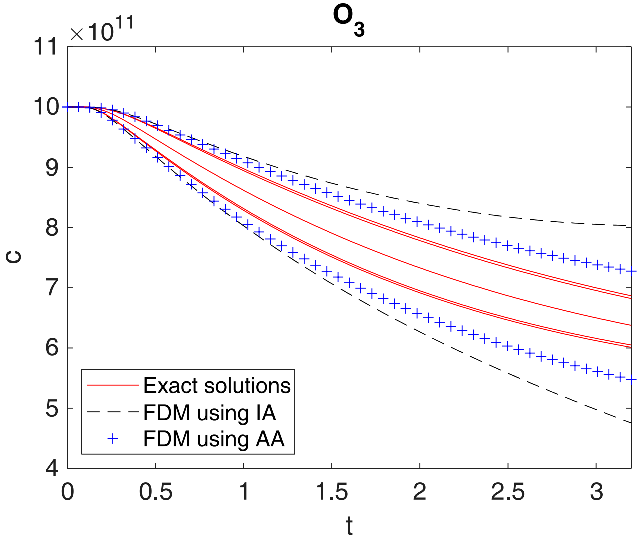
$$O + O_2 \xrightarrow{k_1} O_3$$
$$O + O_3 \xrightarrow{k_2} 2O_2$$
$$O_2 \xrightarrow{k_3(t)} 2O$$
$$O_3 \xrightarrow{k_4(t)} O + O_2$$

$$\frac{\partial c_O}{\partial t} + u_1 \frac{\partial c_O}{\partial y} = D_1 \frac{\partial^2 c_O}{\partial y^2} + r_{c_O},$$
$$\frac{\partial c_{O_2}}{\partial t} + u_2 \frac{\partial c_{O_2}}{\partial y} = D_2 \frac{\partial^2 c_{O_2}}{\partial y^2} + r_{c_{O_2}},$$
$$\frac{\partial c_{O_3}}{\partial t} + u_3 \frac{\partial c_{O_3}}{\partial y} = D_3 \frac{\partial^2 c_{O_3}}{\partial y^2} + r_{c_{O_3}},$$

$$u \in [4\text{e-}3, 6\text{e-}3]$$
$$D \in [4\text{e-}3, 6\text{e-}3]$$







Transport Model in Tumor

- Fluid Transport Model
  - Darcy's law $u = -K\dfrac{dp}{dr}$
  - Continuity equation

$$\nabla \cdot \mathbf{u} = \phi_v(r)$$

$$\phi_v(r) = L_p \frac{S}{V}(p_v - p)$$

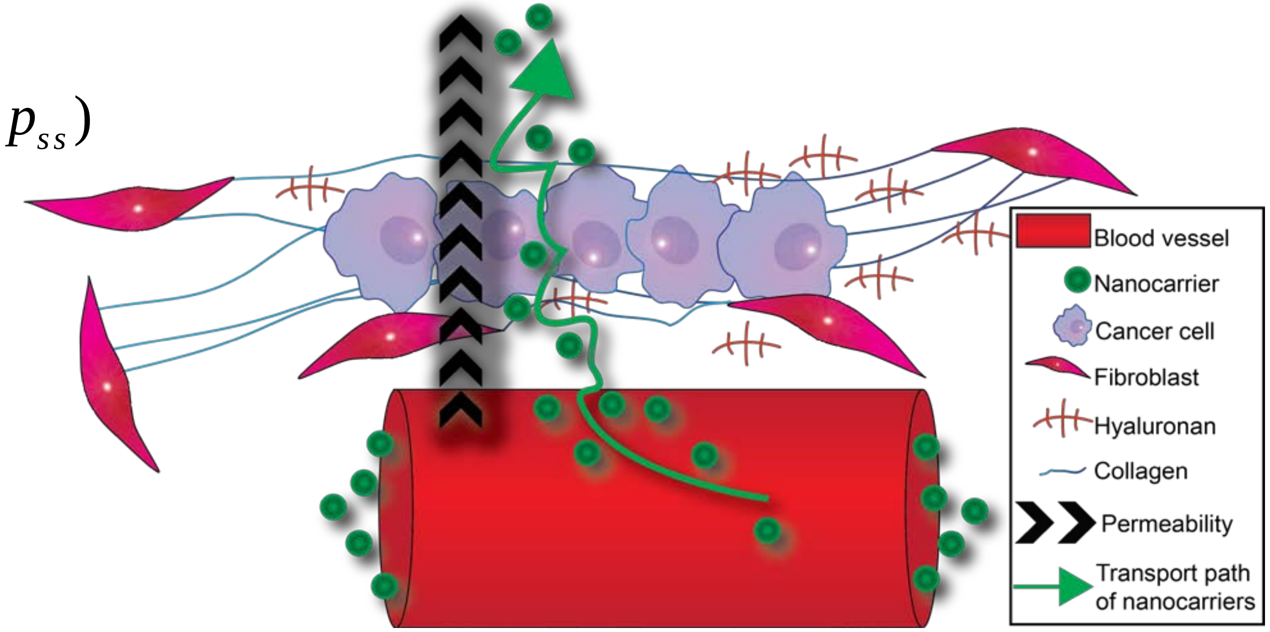
$$\nabla^2 p = \frac{\alpha^2}{R^2}(p - p_{ss})$$

- Solute Transport Model

$$\frac{\partial C}{\partial t} + \underbrace{\frac{1}{r^2}\frac{\partial (r^2 uC)}{\partial r}}_{\text{Convection}} = \underbrace{D\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial C}{\partial r}\right)}_{\text{Diffusion}} + \phi_s$$

- Pore Theory

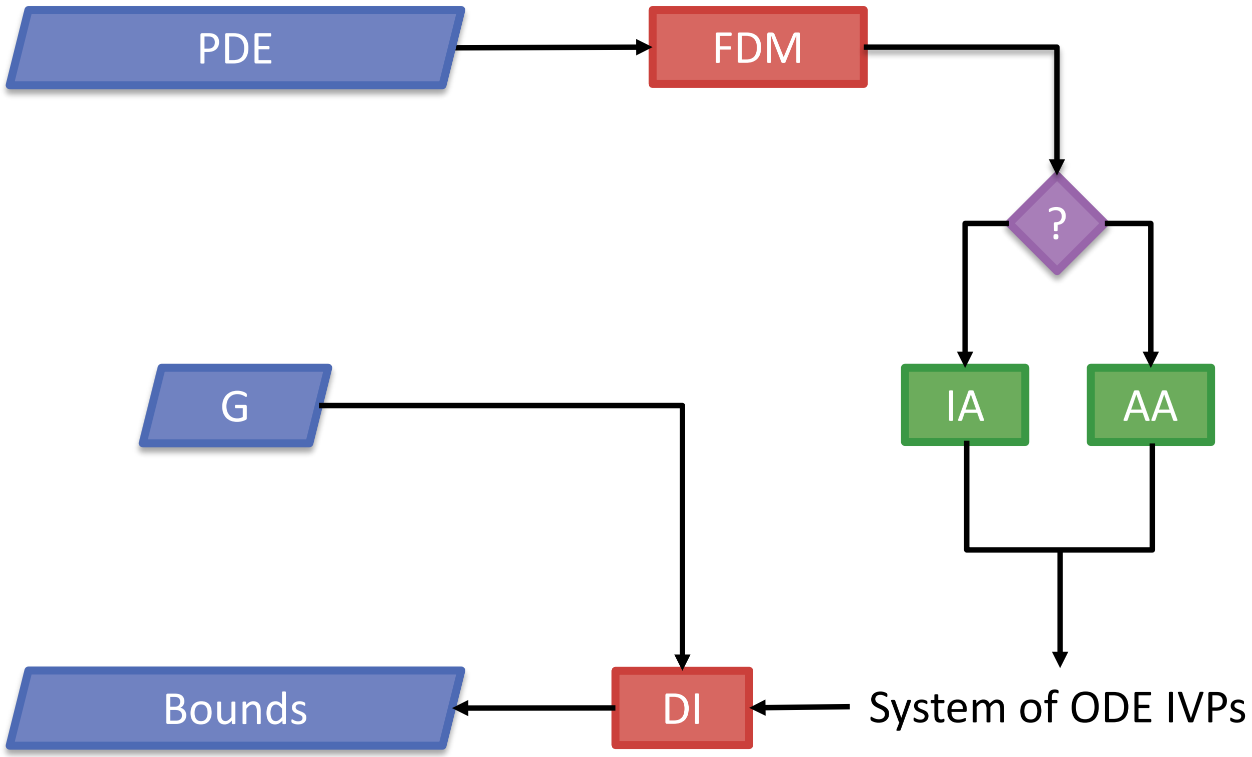
$$L_p = \frac{\gamma r_o^2}{8\mu L} \qquad P = \frac{\gamma H D_o}{L} \qquad \sigma = 1 - W$$



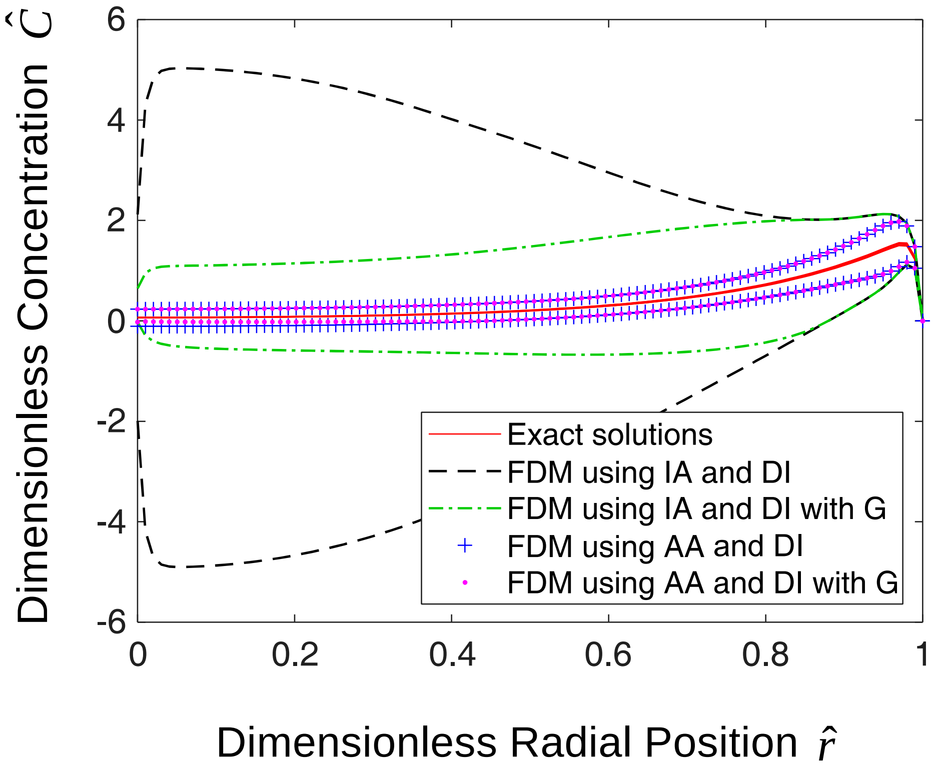
[9] J. D. Martin, M. Panagi, C. Wang, T. T. Khan, M. R. Martin, C. Voutouri, K. Toh, P. Papageorgis, F. Mpekris, C. Polydorou, et al., "Dexamethasone increases cisplatin-loaded nanocarrier delivery and efficacy in metastatic breast cancer by normalizing the tumor microenvironment," *ACS nano*.

Transport Model in Tumor

$K \in [2.75\text{E-}7, 2.85\text{E-}7]$ $L_p \in [1.65\text{E-}6, 1.75\text{E-}6]$

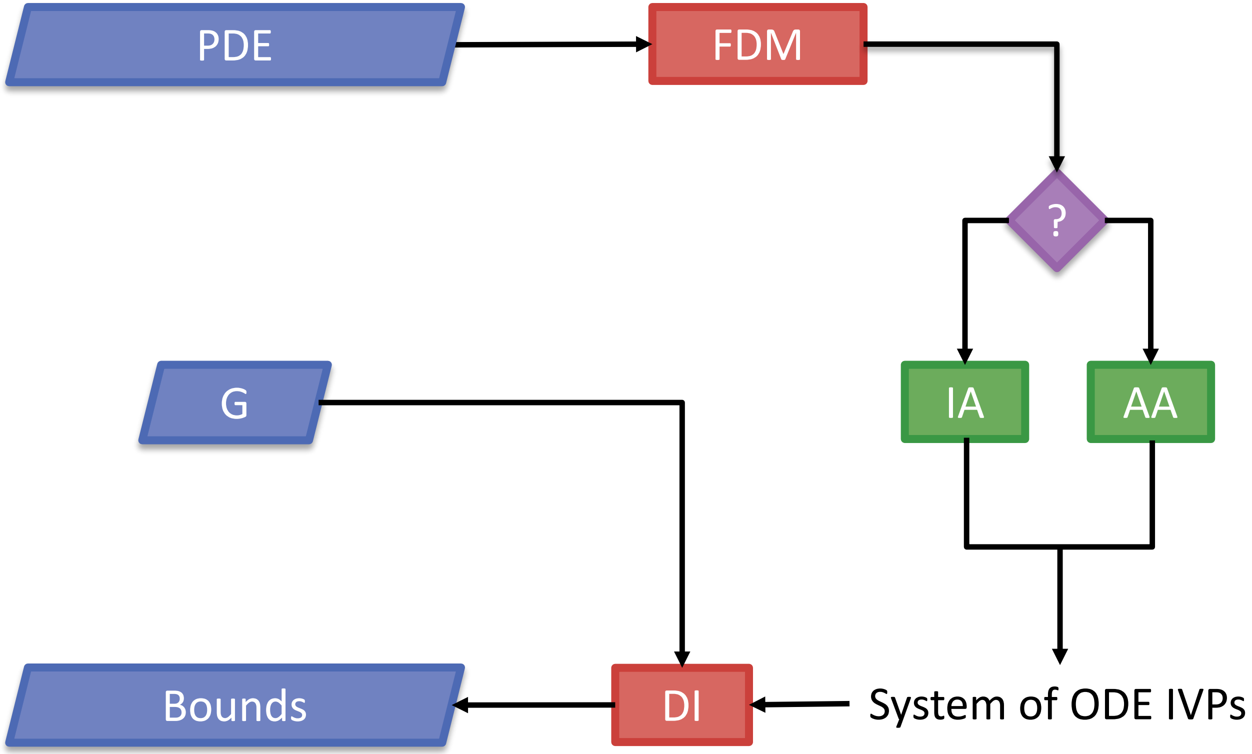


Concentration at 30 min post injection

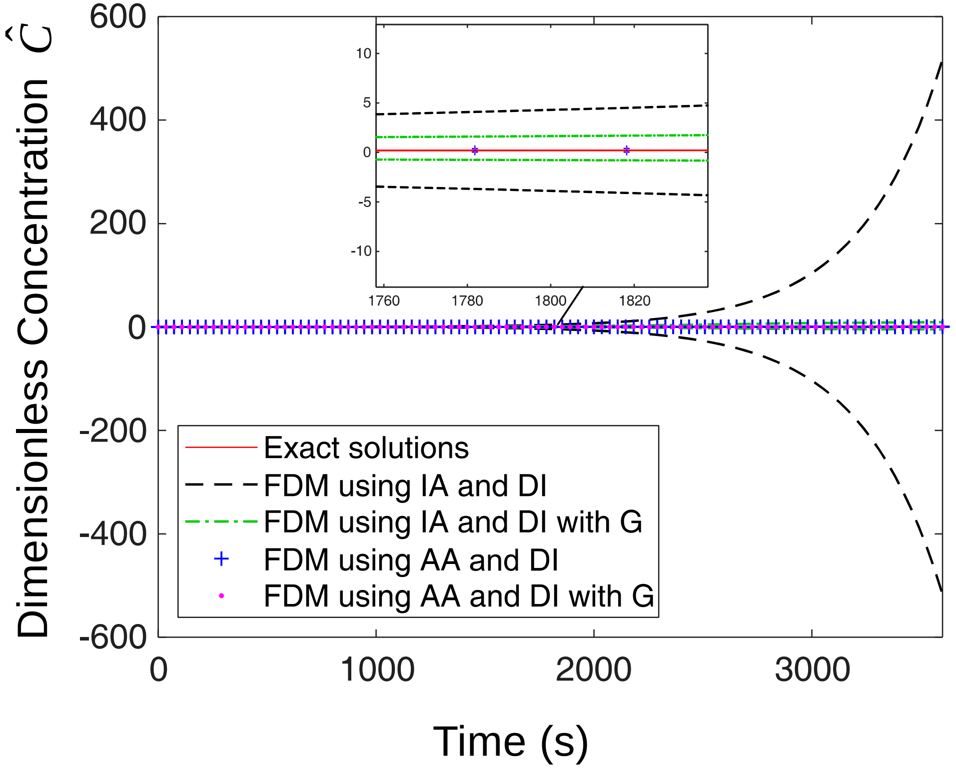


Transport Model in Tumor

$K \in [2.75\text{E-}7, 2.85\text{E-}7] \quad L_p \in [1.65\text{E-}6, 1.75\text{E-}6]$



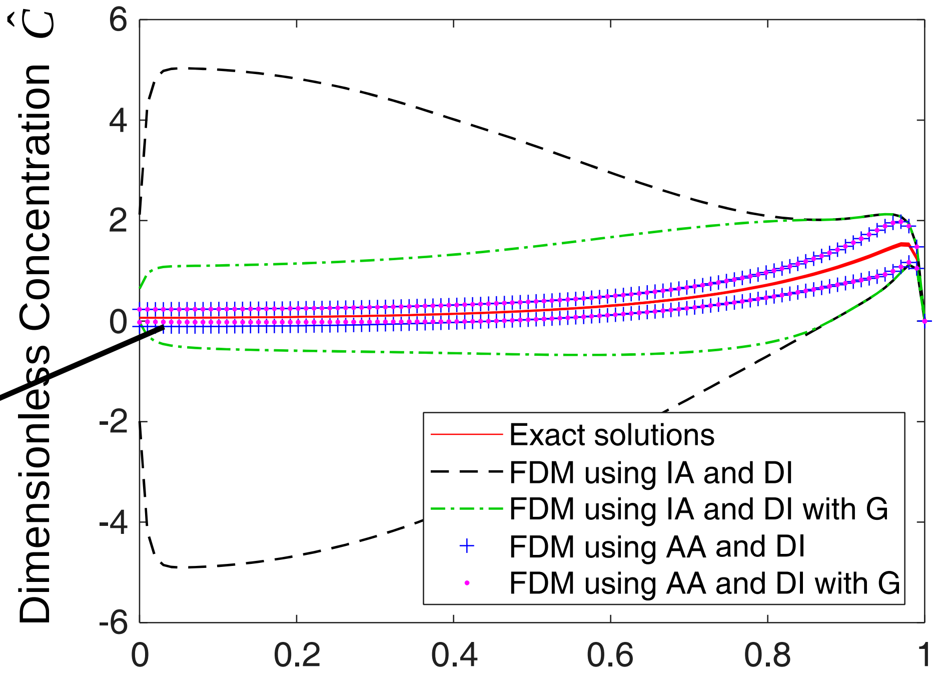
Concentration at medium radius

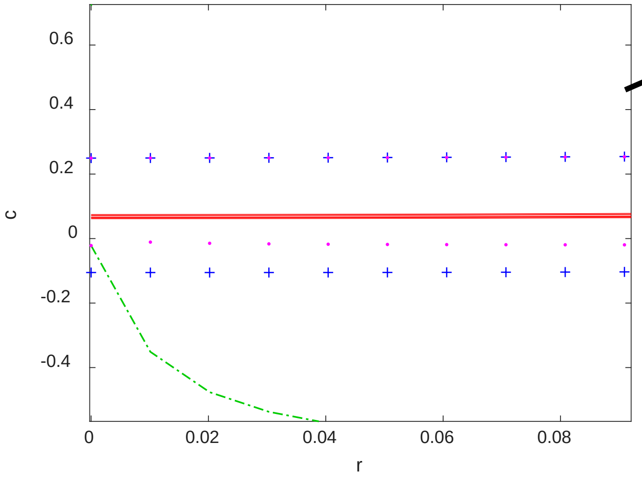


Transport Model in Tumor

Method	Time (s)
Exact solutions	0.156898524
FDM using IA and DI	4.366923814
FDM using IA and DI with G	212.759461536
FDM using AA and DI	4.033772619
FDM using AA and DI with G	201.610427039

Concentration at 30 min post injection





Summary

	Conv-Diff	Conv-Rxn	Diff-Rxn	Tumor Model
FD	$V_{IA}=V_{AA}$	$V_{IA}=V_{AA}$		Unstable
BD	$V_{IA}=V_{AA}$	$V_{IA}=V_{AA}$		$V_{IA}>V_{AA}$
CD	$V_{IA}>V_{AA}$	$V_{IA}>V_{AA}$	$V_{IA}>V_{AA}$	Unstable

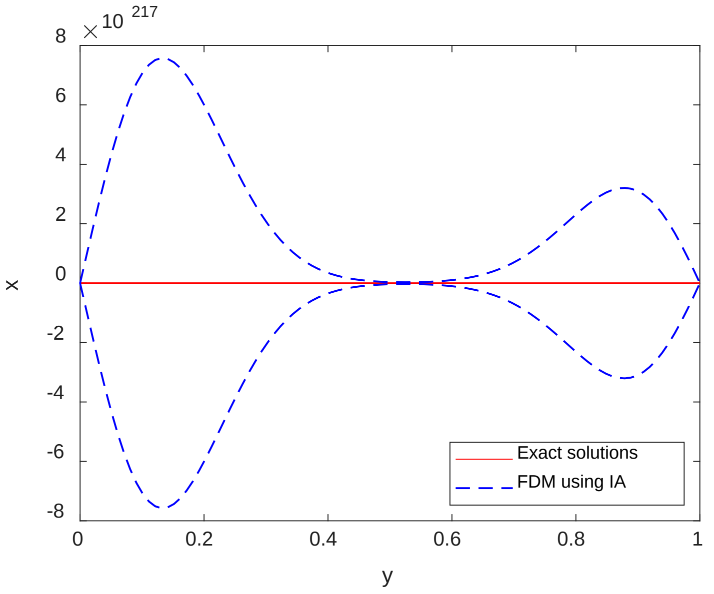
$$\frac{\partial x}{\partial t} = p_1 \frac{\partial^2 x}{\partial y^2} - p_2 \frac{\partial x}{\partial y}, \quad t \in [0,1], \quad y \in [0,1]$$

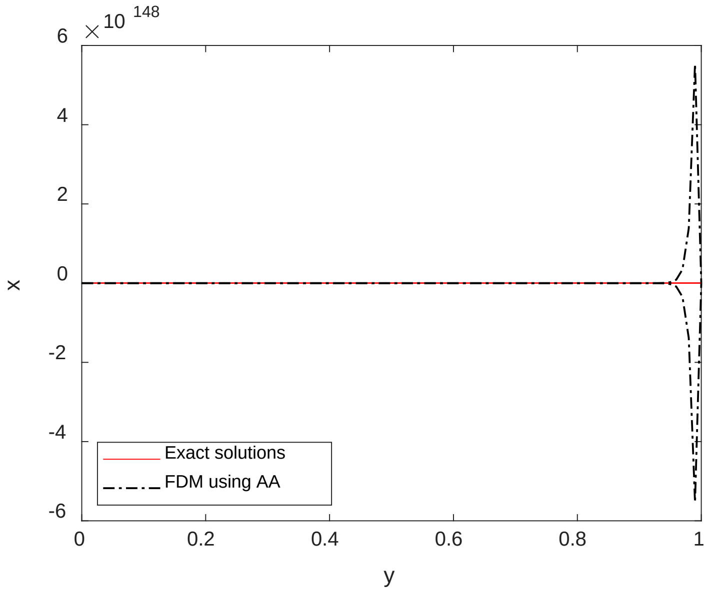
$$p_1 \in [0.1, 0.3] \qquad p_2 \in [0.2, 0.6]$$

Backward finite difference

↓

Interval Extension





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  - Prof. Matthew D. Stuber,
  - Matthew Wilhelm,
  - William Hale
  - Other group members
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  - University of Connecticut
  - National Science Foundation, Award No.: 1560072, 1706343, 1932723

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Any questions?

Process Systems and
Operations Research
Laboratory

Backup Slides

Discrete-time Differential Inequalities

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}(\mathbf{x}(t), \boldsymbol{\pi}(t)),\ \ \mathbf{x}(t_0) = \mathbf{c}_0,\ \ t \in [t_0, t_f]$$

Explicit Euler

$$\mathbf{x}_{k+1} = \mathbf{x}_k + h\mathbf{f}(\mathbf{x}_k, \boldsymbol{\pi}_k),\ \ \mathbf{x}_0 = \mathbf{c}_0,\ \ \forall k \in \{0, \ldots, \kappa\}$$

Interval extension

$$x^L_{k+1,i} = x^L_{k,i} + hf^L_i([\mathbf{x}^L_k, \mathbf{x}^U_k], \Pi),\ \ x^L_{0,i} = c^L_{0,i}$$
$$x^U_{k+1,i} = x^U_{k,i} + hf^U_i([\mathbf{x}^L_k, \mathbf{x}^U_k], \Pi),\ \ x^U_{0,i} = c^U_{0,i}$$

Flattening operators

$$\beta^L_i([\mathbf{x}^L, \mathbf{x}^U]) = \{\mathbf{z} \in [\mathbf{x}^L, \mathbf{x}^U] : z_i = x^L_i\}$$
$$\beta^U_i([\mathbf{x}^L, \mathbf{x}^U]) = \{\mathbf{z} \in [\mathbf{x}^L, \mathbf{x}^U] : z_i = x^U_i\}$$

$$x^L_{k+1,i} = x^L_{k,i} + hf^L_i(\beta^L_i[\mathbf{x}^L_k, \mathbf{x}^U_k], \Pi),\ \ x^L_{0,i} = c^L_{0,i}$$
$$x^U_{k+1,i} = x^U_{k,i} + hf^U_i(\beta^U_i[\mathbf{x}^L_k, \mathbf{x}^U_k], \Pi),\ \ x^U_{0,i} = c^U_{0,i}$$

Prior enclosure G →

$$x^L_{k+1,i} = x^L_{k,i} + hf^L_i(\mathcal{I}_G(\beta^L_i([\mathbf{x}^L_k, \mathbf{x}^U_k])), \Pi),\ \ \ x^L_{0,i} = c^L_{0,i}$$
$$x^U_{k+1,i} = x^U_{k,i} + hf^U_i(\mathcal{I}_G(\beta^U_i([\mathbf{x}^L_k, \mathbf{x}^U_k])), \Pi),\ \ \ x^U_{0,i} = c^U_{0,i}$$

- For continuous-time system, a trajectory $\mathbf{x}(t; \mathbf{c}_0, \boldsymbol{\pi})$ cannot leave the enclosure $[\mathbf{x}^L(t), \mathbf{x}^U(t)]$ without crossing its boundary
- Thus, at any $t \in [t_0, t_f]$, it is only necessary for $x^L_i(t)$ to decrease faster than all trajectories $x_i(t; \mathbf{c}_0, \boldsymbol{\pi})$ that are already incident on the i$^{\text{th}}$ lower face of $[\mathbf{x}^L(t), \mathbf{x}^U(t)]$.

Example: $[\mathbf{x}^L, \mathbf{x}^U] = \{[1,5], [2,4], [1,7]\}$

$$\beta^L_2([\mathbf{x}^L, \mathbf{x}^U]) = \{[1,5], [2,2], [1,7]\}$$
$$\beta^U_2([\mathbf{x}^L, \mathbf{x}^U]) = \{[1,5], [4,4], [1,7]\}$$

I.C. and B.C.

$$\frac{\partial x}{\partial t} = p_1 \frac{\partial^2 x}{\partial y^2} - p_2 \frac{\partial x}{\partial y},\ t \in [0,1],\ y \in [0,1]$$

$$x(y,0) = 2$$

$$x(0,t) = 0,\ x(1,t) = 1$$

$$\frac{\partial x}{\partial t} = -p_1 \frac{\partial x}{\partial y} - p_2 x,\ t \in [0,1],\ y \in [0,1]$$

$$x(y,0) = 1$$

$$x(0,t) = 0,\ \frac{dx}{dy}\Big|_{y=1} = 0$$

$$\frac{\partial x}{\partial t} = p_1 \frac{\partial^2 x}{\partial y^2} - p_2 x,\ t \in [0,1],\ y \in [0,1]$$

$$x(y,0) = 2$$

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Coupled IBVPs

$$O + O_2 \xrightarrow{k_1} O_3$$
$$O + O_3 \xrightarrow{k_2} 2O_2$$
$$O_2 \xrightarrow{k_3(t)} 2O$$
$$O_3 \xrightarrow{k_4(t)} O + O_2$$

$$\frac{\partial c_O}{\partial t} + u_1 \frac{\partial c_O}{\partial y} = D_1 \frac{\partial^2 c_O}{\partial y^2} + r_{c_O},$$
$$\frac{\partial c_{O_2}}{\partial t} + u_2 \frac{\partial c_{O_2}}{\partial y} = D_2 \frac{\partial^2 c_{O_2}}{\partial y^2} + r_{c_{O_2}},$$
$$\frac{\partial c_{O_3}}{\partial t} + u_3 \frac{\partial c_{O_3}}{\partial y} = D_3 \frac{\partial^2 c_{O_3}}{\partial y^2} + r_{c_{O_3}},$$

$$u \in [\text{4e-3,6e-3}]$$
$$D \in [\text{4e-3,6e-3}]$$

$$r_{c_O} = \frac{d_{c_O}}{dt} = -k_1 c_O c_{O_2} - k_2 c_O c_{O_3} + 2k_3(t) c_{O_2} + k_4(t) c_{O_3},$$
$$r_{c_{O_2}} = \frac{d_{c_{O_2}}}{dt} = -k_1 c_O c_{O_2} + k_2 c_O c_{O_3} - k_3(t) c_{O_2} + k_4(t) c_{O_3},$$
$$r_{c_{O_3}} = \frac{d_{c_{O_3}}}{dt} = k_1 c_O c_{O_2} - k_2 c_O c_{O_3} - k_4(t) c_{O_3}.$$

[8] J. Makungu, H. Haario, W. C. Mahera, “A generalized 1-dimensional particle transport method for convection diffusion reaction model,” *Afrika Matematika*, 23 (1) (2012) 21–39.

Coupled IBVPs

Here, the rate constants k_1 and k_2 are constants: $k_1 = 1.63 \times 10^{-16}$, $k_2 = 4.66 \times 10^{-16}$. The other two rate constants $k_3(t)$ and $k_4(t)$ follow a two-day periodical cycle as:

$$k_i(t) = \begin{cases} \exp(-c_i/\sin(\omega t)) & \text{if } \sin(\omega t) \geq 0 \\ 0 & \text{if } \sin(\omega t) > 0, \quad i = 3, 4, \end{cases}$$

where $c_3 = 22.62$, $c_4 = 7.601$, and $\omega = \pi/43200\ \text{s}^{-1}$. The rate constants k_3 and k_4 will increase quickly at the beginning ($t = 0$), rise to a peak at noon ($t = 6 \times 3600$ s), then drop to zero at sunset ($t = 12 \times 3600$ s). The initial conditions of this ozone model is given by

$$c_O(x,0) = \begin{cases} 10^6 & \text{if } 0.3 \leq x \leq 0.6 \\ 0 & \text{otherwise,} \end{cases}$$

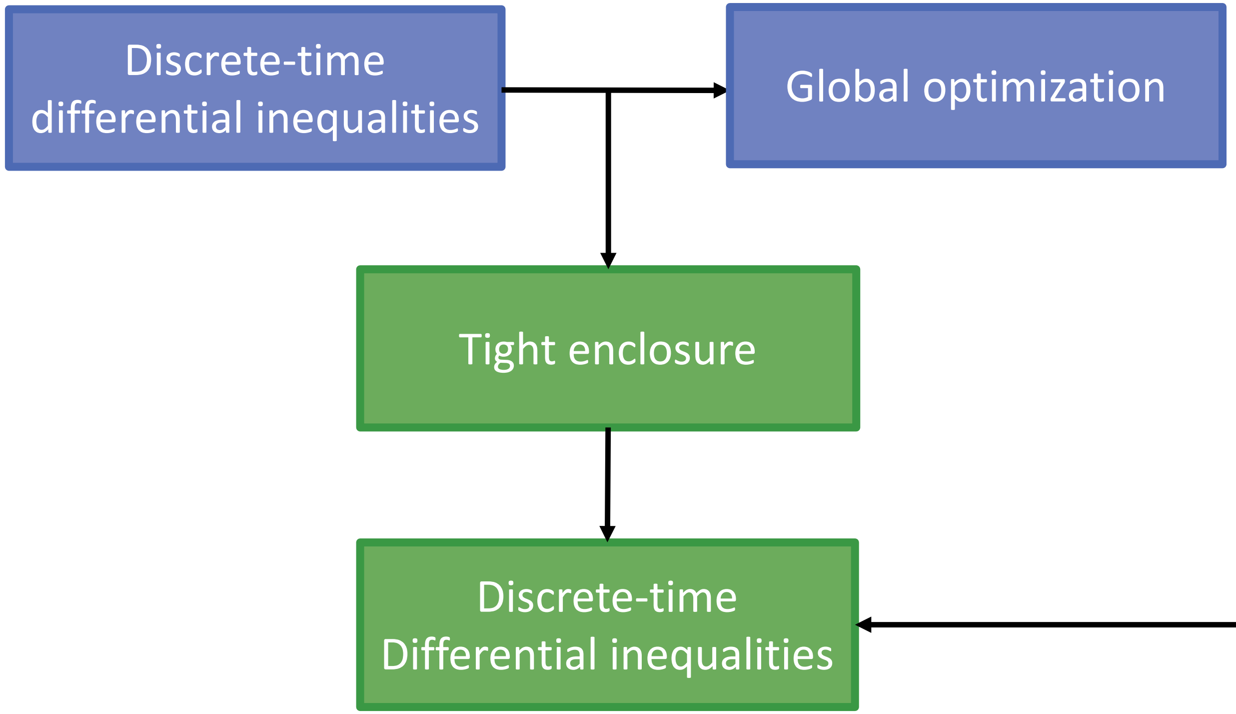
$$c_{O_2}(x,0) = \begin{cases} 3.7 \times 10^{16} & \text{if } 0.3 \leq x \leq 0.6 \\ 0 & \text{otherwise,} \end{cases}$$

$$c_{O_3}(x,0) = \begin{cases} 10^{12} & \text{if } 0.3 \leq x \leq 0.6 \\ 0 & \text{otherwise.} \end{cases}$$

At the bottom of the atmosphere layer, the boundary conditions are given by $c_O(0,t) = 10^6$, $c_{O_2}(0,t) = 3.7 \times 10^{16}$, $c_{O_3}(0,t) = 10^6$, while at the top of the atmosphere, we assume the no-flux boundary conditions given by $\left.\frac{dc_O}{dy}\right|_{y=1} = \left.\frac{dc_{O_2}}{dy}\right|_{y=1} = \left.\frac{dc_{O_3}}{dy}\right|_{y=1} = 0$.

[8] J. Makungu, H. Haario, W. C. Mahera, "A generalized 1-dimensional particle transport method for convection diffusion reaction model," *Afrika Matematika*, 23 (1) (2012) 21–39.

Discrete-time Differential Inequalities



$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}(\mathbf{x}(t),\boldsymbol{\pi}(t)),\ \mathbf{x}(t_0) = \mathbf{c}_0,\ t \in [t_0, t_f]$$

Explicit Euler

$$\mathbf{x}_{k+1} = \mathbf{x}_k + h\mathbf{f}(\mathbf{x}_k,\boldsymbol{\pi}_k),\ \mathbf{x}_0 = \mathbf{c}_0,\ \forall k \in \{0,\dots,\kappa\}$$

Interval extension

$$x^L_{k+1,i} = x^L_{k,i} + hf^L_i([\mathbf{x}^L_k,\mathbf{x}^U_k],\Pi),\ x^L_{0,i} = c^L_{0,i}$$
$$x^U_{k+1,i} = x^U_{k,i} + hf^U_i([\mathbf{x}^L_k,\mathbf{x}^U_k],\Pi),\ x^U_{0,i} = c^U_{0,i}$$

Flattening operators

$$\beta^L_i([\mathbf{x}^L,\mathbf{x}^U]) = \{\mathbf{z} \in [\mathbf{x}^L,\mathbf{x}^U] : z_i = x^L_i\}$$
$$\beta^U_i([\mathbf{x}^L,\mathbf{x}^U]) = \{\mathbf{z} \in [\mathbf{x}^L,\mathbf{x}^U] : z_i = x^U_i\}$$

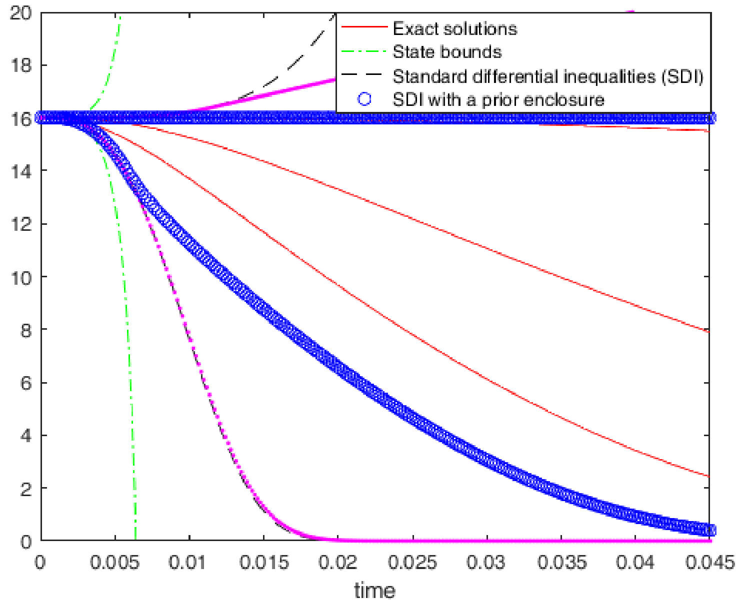
$$x^L_{k+1,i} = x^L_{k,i} + hf^L_i(\beta^L_i[\mathbf{x}^L_k,\mathbf{x}^U_k],\Pi),\ x^L_{0,i} = c^L_{0,i}$$
$$x^U_{k+1,i} = x^U_{k,i} + hf^U_i(\beta^U_i[\mathbf{x}^L_k,\mathbf{x}^U_k],\Pi),\ x^U_{0,i} = c^U_{0,i}$$

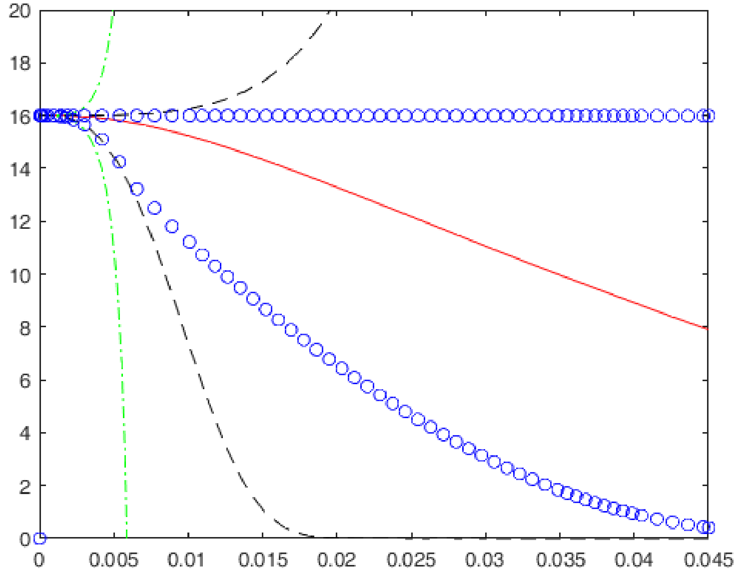
[9] X. Yang and J. K. Scott, "Efficient reachability bounds for discrete-time nonlinear systems by extending the continuous-time theory of differential inequalities," in 2018 Annual American Control Conference (ACC), pp. 6242-6247, IEEE, 2018.

Discrete-time DI vs Continuous-time DI

Discrete-time methods	Time per step	Time
exact	0.0001506623	0.035599025
Interval	0.001027332844	0.241191195
SDI	0.000212136452	0.048447647
IGB	0.0005043008	0.122923914
BIG	0.000196413496	0.052653721
MG	0.000389502404	0.096926013

Continuous-time methods	Time	Steps	Time per step	Literature
exact	0.167765429	44	0.0038128507	0.004
Interval	0.28842996	63	0.0045782533	
SDI	0.374489244	1160	0.0003228356	0.014
MG	0.097186758	64	0.0015185431	0.055





Hukuhara = Affine

$$X = [x^L, x^U], Y = [y^L, y^U]$$

$$m_X = \frac{x^L + x^U}{2}, r_X = \frac{x^U - x^L}{2}, m_Y = \frac{y^L + y^U}{2}, r_Y = \frac{y^U - y^L}{2}$$

$$\overline{X} = m_X + r_X \epsilon$$

$$\overline{Y} = m_Y + r_Y \epsilon$$

when $r_X \geq r_Y$

$$\begin{aligned}
\overline{X} - \overline{Y} &= (m_X + r_X \epsilon) - (m_Y + r_Y \epsilon) \\
&= (m_X - m_Y) + (r_X - r_Y)\epsilon \\
&= \left(\frac{x^L + x^U}{2} - \frac{y^L + y^U}{2}\right) + \left(\frac{x^U - x^L}{2} - \frac{y^U - y^L}{2}\right)\epsilon \\
&= \frac{1}{2}(x^L - y^L + x^U - y^U) + \frac{1}{2}(x^U - y^U - x^L + y^L)[-1,1] \\
&= \frac{1}{2}(x^L - y^L + x^U - y^U) + \frac{1}{2}[-x^U + y^U + x^L - y^L, x^U - y^U - x^L + y^L] \\
&= [x^L - y^L, x^U - y^U]
\end{aligned}$$

when $r_X < r_Y$

$$\begin{aligned}
\overline{X} - \overline{Y} &= (m_X + r_X \epsilon) - (m_Y + r_Y \epsilon) \\
&= (m_X - m_Y) + (r_X - r_Y)\epsilon \\
&= \left(\frac{x^L + x^U}{2} - \frac{y^L + y^U}{2}\right) + \left(\frac{x^U - x^L}{2} - \frac{y^U - y^L}{2}\right)\epsilon \\
&= \frac{1}{2}(x^L - y^L + x^U - y^U) + \frac{1}{2}(x^U - y^U - x^L + y^L)[-1,1] \\
&= \frac{1}{2}(x^L - y^L + x^U - y^U) + \frac{1}{2}[x^U - y^U - x^L + y^L, -x^U + y^U + x^L - y^L] \\
&= [x^U - y^U, x^L - y^L]
\end{aligned}$$