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Approaches to Improve Bilinear Relaxations in Reduced-Space

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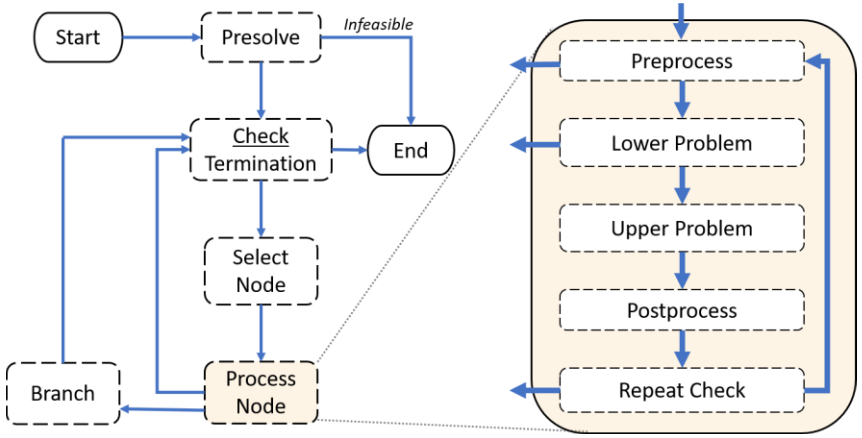
Process Systems and Operations Research Laboratory

Global Optimization

- Nonconvex MINLP formulations naturally arise in many applications.
- MINLP solvers generally rely on some variation of spatial branch-and-bound[2,3].
- Relaxed subproblems are used to compute bounds and are often derived from relaxed functions[2,3].



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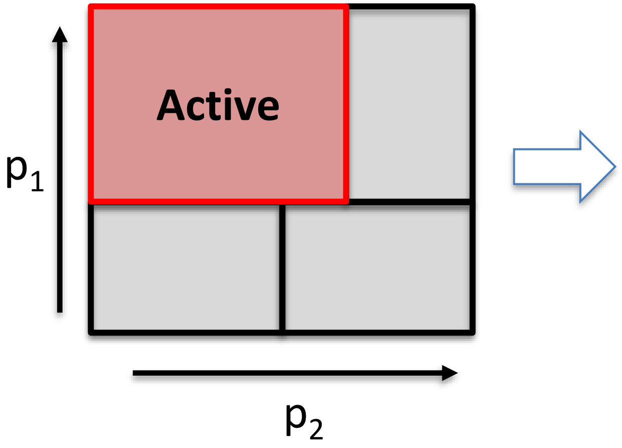
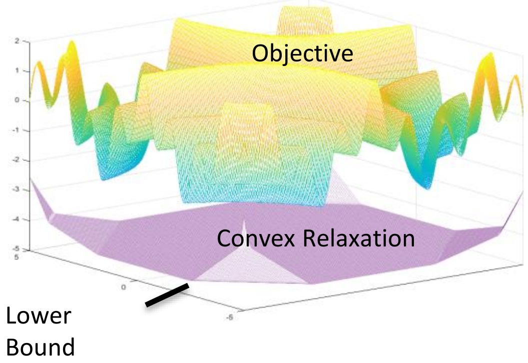


(Relax Problem + Compute Bound)



1. Wilhelm, M.E., and Stuber, M.D.. **EAGO.jl: easy advanced global optimization in Julia.** *Optimization Methods and Software*, 1-26.
2. Horst, Reiner, and Hoang Tuy. *Global optimization: Deterministic approaches*. Springer Science & Business Media, 2013.
3. Puranik, Yash, and Nikolaos V. Sahinidis. **Domain reduction techniques for global NLP and MINLP optimization.** *Constraints* 22.3 (2017): 338-376.

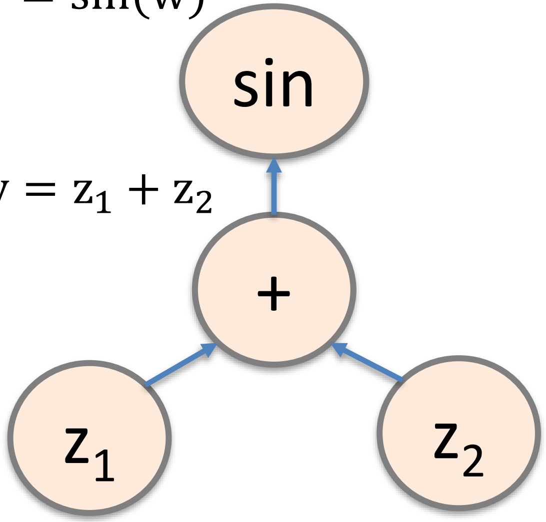
Reduced-Space Relaxations

Factorable Function[4]

$$y = \sin(z_1 + z_2)$$

$y = \sin(w)$

$w = z_1 + z_2$



Compute $y^{cv}(\mathbf{z}), y^{cc}(\mathbf{z}), y^L, y^U$

Compute $w^{cv}(\mathbf{z}), w^{cc}(\mathbf{z}), w^L, w^U$

Initialize $\mathbf{z}^{cv}, \mathbf{z}^{cc}, \mathbf{z}^L, \mathbf{z}^U$

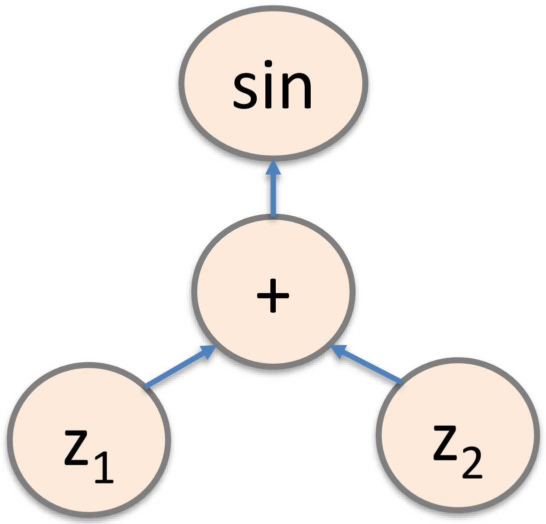
Notation

$q^{cv}(\mathbf{z})$	**Convex relaxation**
$q^{cc}(\mathbf{z})$	**Concave relaxation**
q^L	**Lower bound**
q^U	**Upper bound**

4. Mitsos, A, et al. **McCormick-based relaxations of algorithms.** *SIAM Journal on Optimization*, 20, (2009): 573 - 601.

Reduced-Space Relaxations

Factorable Function[4]



PROS

- Tight relaxations
- Favorable convergence properties
- Relaxed problem same dimension as decision space
- Ease of implementation

Repeatedly enclosing nonconvex functions with convex/concave functions.

CONS

- Multidimensional.
- Nonsmooth
- Overly conservative.

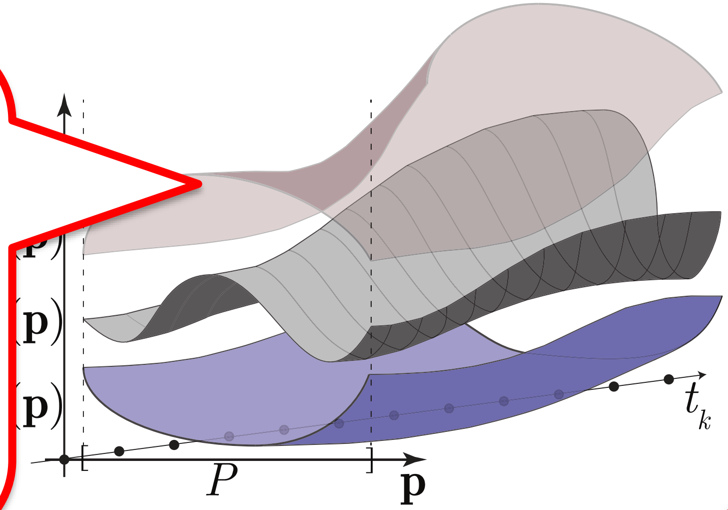
4. Mitsos, A, et al. **McCormick-based relaxations of algorithms.** *SIAM Journal on Optimization*, 20, (2009): 573 - 601.

Reduced-Space Relaxations

Factorable Function[4]

- ❖ Auxiliary ODEs or discretize-and-relax
- ❖ Repeated relaxation of right-hand-side function

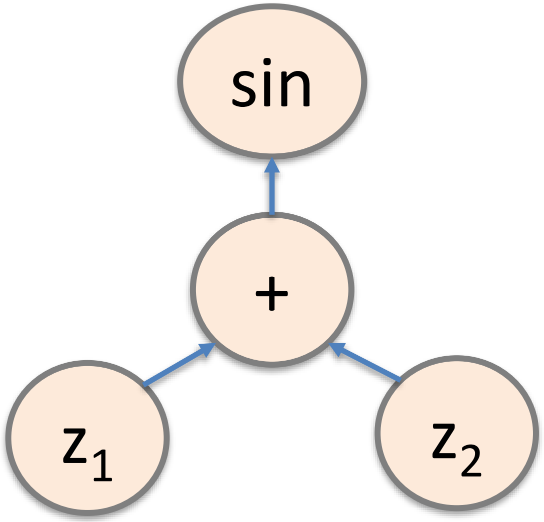
Parametric ODEs[5,6,7]



4. Mitsos, A, et al. **McCormick-based relaxations of algorithms.** *SIAM Journal on Optimization*, 20, (2009): 573 - 601.
5. Wilhelm, ME; Le, AV; and Stuber. MD. **Global Optimization of Stiff Dynamical Systems.** *AIChE Journal: Futures Issue,* 65 (12), *2019.*
6. Sahlodin, A.M., and Chachuat, B. **Discretize-then-relax approach for convex/concave relaxations of the solutions of parametric ODEs.** *Applied Numerical Mathematics* 61.7 (2011): 803-820
7. Scott, Joseph K., and Paul I. Barton. **Improved relaxations for the parametric solutions of ODEs using differential inequalities.** *Journal of Global Optimization* 57.1 (2013): 143-176.

Reduced-Space Relaxations

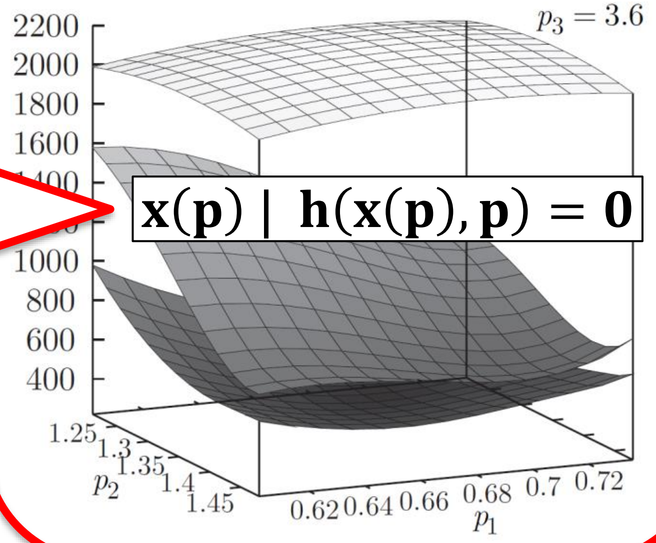
Factorable Function[4]



Parametric ODEs[5,6,7]

- A fixed-point method, involves operation similar linear solve.
- Repeated addition, subtraction, multiplication.

Implicit Functions[8]



4. Mitsos, A, et al. **McCormick-based relaxations of algorithms.** *SIAM Journal on Optimization*, 20, (2009): 573 - 601.
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8. Stuber, MD et al. **Convex and concave relaxations of implicit functions.** *Optimization Methods and Software* 30, (2015), 424-460

Reduced-Space Relaxations

Select Contributions to McCormick Relaxation Theory

a) Special treatment of specific functional forms[10]
b) Improvements to Multivariate Relaxations[11]
c) Differentiable Relaxations[12]

10. D Bongartz, J Najman, A Mitsos. **Deterministic global optimization of steam cycles using the IAPWS-IF97 model.** *Optimization and Engineering* 21, (2020): 1095-1131.
11. Tsoukalas, A., Mitsos, A. **Multivariate McCormick Relaxations**. *Journal of Global Optimization* 59, (2014): 633 - 662.
12. Khan, K. et al. **Differentiable McCormick relaxations**. *Journal Global Optimization* 67, (2017): 687-729.

A Priori Estimators

Underestimators u_x and u_y of *x* and *y* are known such that

$$x^L \le u_x \le \min(x, a_x),\ x \le x^U$$
$$y^L \le u_y \le \min(y, a_y),\ y \le y^U$$

Taotao He, Mohit Twarmalini. **A new framework to relax composite functions in nonlinear programs**. *Mathematical Programming*, **190**, pages 427–466 (2021).

4x overestimating inequalities

$$w \ge (y^U - y^L)u_x + y^L x + a_x y - a_x y^U$$

$$w \ge (x^U - x^L)u_y + a_y x + x^L y - a_y x^U$$

$$w \ge (a_y - y^L)u_x + (a_x - x^L)u_y + y^L x + x^L y - a_x a_y$$

$$w \ge (y^U - a_y)u_x + (x^U - a_x)u_y + a_y x + a_x y + a_x a_y - a_x y^U - x^U a_y$$

+

8x (under/over)-estimators from over-estimators v_x, v_y with upper bounds b_x, b_y

$$w = (-x)(-y)$$

Relaxation of Bilinear Term

McCormick Relaxation of Bilinear Term[13]

$$w = x \times y$$

Underestimators

$$(1) \quad w \geq x^L y + x y^L - x^L y^L$$

$$(2) \quad w \geq x^U y + x y^U - x^U y^U$$

Overestimators

$$(1) \quad w \leq x^U y + x y^L - x^U y^L$$

$$(2) \quad w \leq x y^U + x^L y - x^L y^U$$

13. McCormick, Garth P. **Computability of global solutions to factorable nonconvex programs: Part I - Convex underestimating problems.** *Mathematical programming* 10.1, (1976): 147-175.

Relaxation of Composite Bilinear Term

McCormick Relaxation of Composite Bilinear Term[4] $\mathrm{w}(\mathbf{z}) = \mathrm{x}(\mathbf{z}) \times \mathrm{y}(\mathbf{z})$

~~Underestimators~~ Convex Relaxations

$$\mathrm{w}_1^{\mathrm{cv}}(\mathbf{z}) = \min(\mathrm{x}^{\mathrm{L}}\mathrm{y}^{\mathrm{cv}}(\mathbf{z}), \mathrm{x}^{\mathrm{L}}\mathrm{y}^{\mathrm{cc}}(\mathbf{z})) + \min(\mathrm{y}^{\mathrm{L}}\mathrm{x}^{\mathrm{cv}}(\mathbf{z}), \mathrm{y}^{\mathrm{L}}\mathrm{x}^{\mathrm{cc}}(\mathbf{z})) - \mathrm{x}^{\mathrm{L}}\mathrm{y}^{\mathrm{L}}$$

$$\mathrm{w}_2^{\mathrm{cv}}(\mathbf{z}) = \min(\mathrm{x}^{\mathrm{U}}\mathrm{y}^{\mathrm{cv}}(\mathbf{z}), \mathrm{x}^{\mathrm{U}}\mathrm{y}^{\mathrm{cc}}(\mathbf{z})) + \min(\mathrm{y}^{\mathrm{U}}\mathrm{x}^{\mathrm{cv}}(\mathbf{z}), \mathrm{y}^{\mathrm{U}}\mathrm{x}^{\mathrm{cc}}(\mathbf{z})) - \mathrm{x}^{\mathrm{U}}\mathrm{y}^{\mathrm{U}}$$

~~Overestimators~~ Concave Relaxations

$$\mathrm{w}_1^{\mathrm{cc}}(\mathbf{z}) = \max(\mathrm{x}^{\mathrm{U}}\mathrm{y}^{\mathrm{cv}}(\mathbf{z}), \mathrm{x}^{\mathrm{U}}\mathrm{y}^{\mathrm{cc}}(\mathbf{z})) + \max(\mathrm{y}^{\mathrm{L}}\mathrm{x}^{\mathrm{cv}}(\mathbf{z}), \mathrm{y}^{\mathrm{L}}\mathrm{x}^{\mathrm{cc}}(\mathbf{z})) - \mathrm{x}^{\mathrm{U}}\mathrm{y}^{\mathrm{L}}$$

$$\mathrm{w}_2^{\mathrm{cc}}(\mathbf{z}) = \max(\mathrm{y}^{\mathrm{U}}\mathrm{x}^{\mathrm{cv}}(\mathbf{z}), \mathrm{y}^{\mathrm{U}}\mathrm{x}^{\mathrm{cc}}(\mathbf{z})) + \max(\mathrm{x}^{\mathrm{L}}\mathrm{y}^{\mathrm{cv}}(\mathbf{z}), \mathrm{x}^{\mathrm{L}}\mathrm{y}^{\mathrm{cc}}(\mathbf{z})) - \mathrm{x}^{\mathrm{L}}\mathrm{y}^{\mathrm{U}}$$

4. Mitsos, A, et al. **McCormick-based relaxations of algorithms.** *SIAM Journal on Optimization*, 20, (2009): 573 - 601.
13. McCormick, Garth P. **Computability of global solutions to factorable nonconvex programs: Part I - Convex underestimating problems.** *Mathematical programming* 10.1, (1976): 147-175.

A Priori Estimators, Reduced-Space

Relaxations of <u>Composite</u> Bilinear Term A Priori

$$\mathrm{w}(\mathbf{z}) = \mathrm{x}(\mathbf{z}) \times \mathrm{y}(\mathbf{z})$$

$$w_1^{cv}(\mathbf{z}) = \left(y^U - a_y\right)u_x^{cv}(\mathbf{z}) + (x^U - a_x)u_y^{cv}(\mathbf{z}) + \min(a_y x^{cv}(\mathbf{z}), a_y x^{cc}(\mathbf{z})) + \min(a_x y^{cv}(\mathbf{z}), a_x y^{cc}(\mathbf{z})) + a_x a_y - a_x y^U - x^U a_y$$

$$w_2^{cv}(\mathbf{z}) = \left(a_y - y^L\right)u_x^{cv}(\mathbf{z}) + (a_x - x^L)u_y^{cv}(\mathbf{z}) + \min(y^L x^{cv}(\mathbf{z}), y^L x^{cc}(\mathbf{z})) + \min(y^L x^{cv}(\mathbf{z}), y^L x^{cc}(\mathbf{z})) - a_x a_y$$

$$w_3^{cv}(\mathbf{z}) = (y^U - y^L)u_x^{cv}(\mathbf{z}) + \min(y^L x^{cv}(\mathbf{z}), y^L x^{cc}(\mathbf{z})) + \min(a_x y^{cv}(\mathbf{z}), a_x y^{cc}(\mathbf{z})) - a_x y^U$$

$$w_4^{cv}(\mathbf{z}) = (x^U - x^L)u_y^{cv}(\mathbf{z}) + \min(a_y x^{cv}(\mathbf{z}), a_y x^{cc}(\mathbf{z})) + \min(y^L x^{cv}(\mathbf{z}), y^L x^{cc}(\mathbf{z})) - a_y x^U$$

+ 12 additional inequalities

Requires convex/concave relaxations of some a priori under/over-estimators and respective extrema:

$$u_x^{cv}(\mathbf{z}), u_y^{cv}(\mathbf{z}), a_x, a_y, v_x^{cc}(\mathbf{z}), v_y^{cc}(\mathbf{z}), b_x, b_y$$

A Priori Estimators, Reduced Space

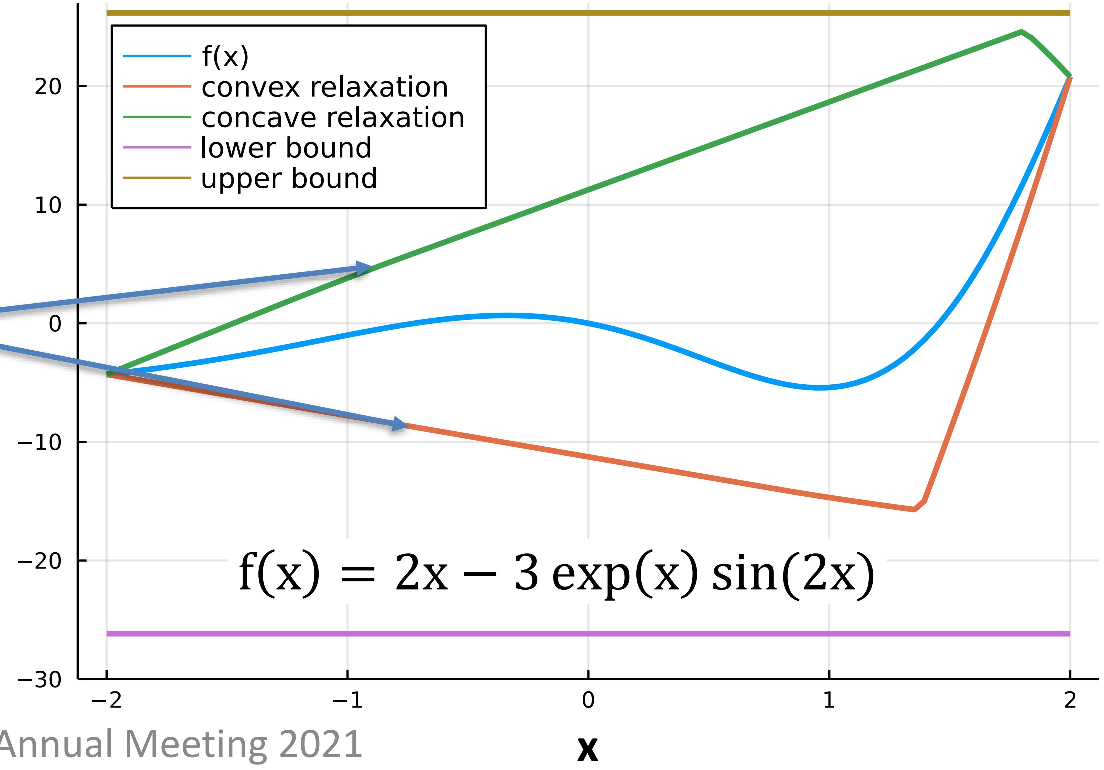
Where to get convex/concave relaxations of a priori estimators (and extremal values)?

Approach 1: McCormick Relaxation

Where to get convex/concave relaxations of a priori estimators (and extremal values)?

Use convex/concave **McCormick relaxations**[4] for under/overestimators

$$x(\mathbf{z}) \geq x^{cv}(\mathbf{z}) = u_x(\mathbf{z}) = u_x^{cv}(\mathbf{z})$$
$$x(\mathbf{z}) \leq x^{cc}(\mathbf{z}) = u_x(\mathbf{z}) = u_x^{cc}(\mathbf{z})$$
$$y(\mathbf{z}) \geq y^{cv}(\mathbf{z}) = u_y(\mathbf{z}) = u_y^{cv}(\mathbf{z})$$
$$y(\mathbf{z}) \leq y^{cc}(\mathbf{z}) = u_y(\mathbf{z}) = u_y^{cc}(\mathbf{z})$$



$$f(x) = 2x - 3\exp(x)\sin(2x)$$

4. Mitsos, A, et al. **McCormick-based relaxations of algorithms.** *SIAM Journal on Optimization*, 20, (2009): 573 - 601.

Approach 1: McCormick Relaxation

Where to get convex/concave relaxations of a priori estimators (and extremal values)?

Us…

r… …s[4]

$x($…
$x($…
$y($…
$y($…

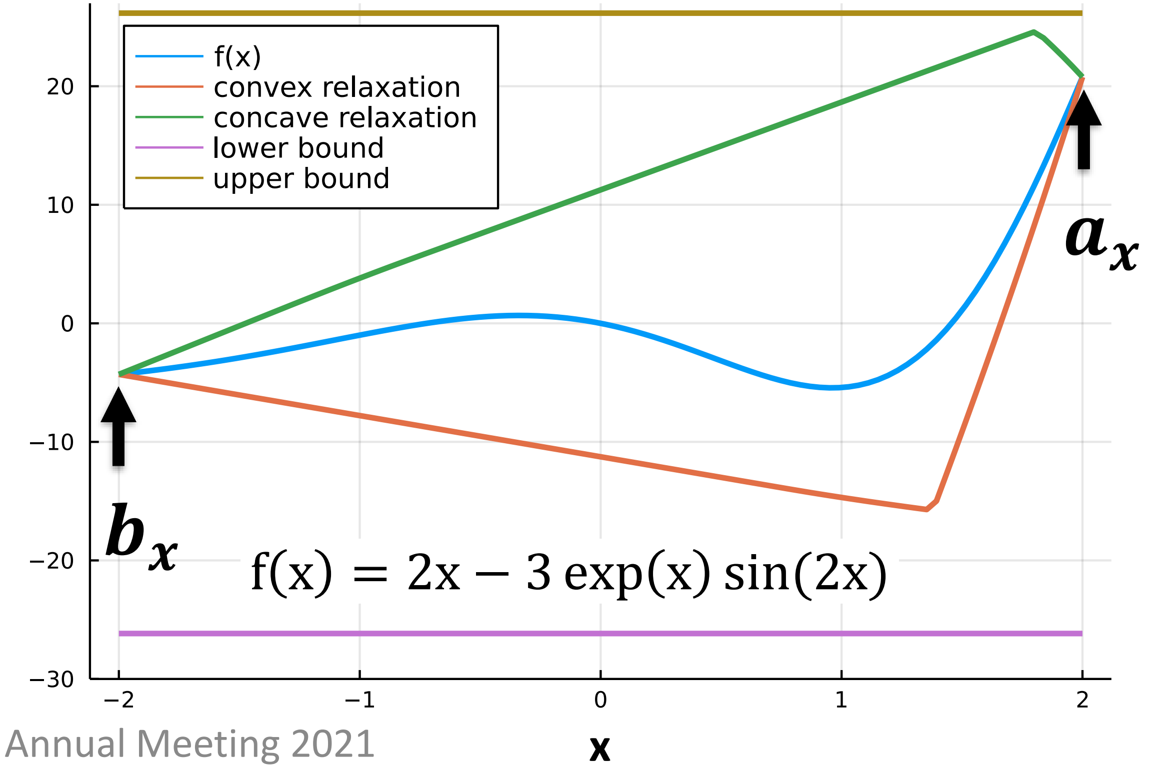
Concave optimization problems over box Z[14]

$$a_x \geq \max_{z \in Z} x^{cv}(\mathbf{z})$$

$$b_x \leq \min_{z \in Z} x^{cc}(\mathbf{z})$$

$$a_y \geq \max_{z \in Z} y^{cv}(\mathbf{z})$$

$$b_x \leq \min_{z \in Z} y^{cc}(\mathbf{z})$$



4. M… J…

14. Pa… mini… …e …): 367-379.

Approach 2: Subgradients

Where to get convex/concave relaxations of a priori estimators (and extremal values)?

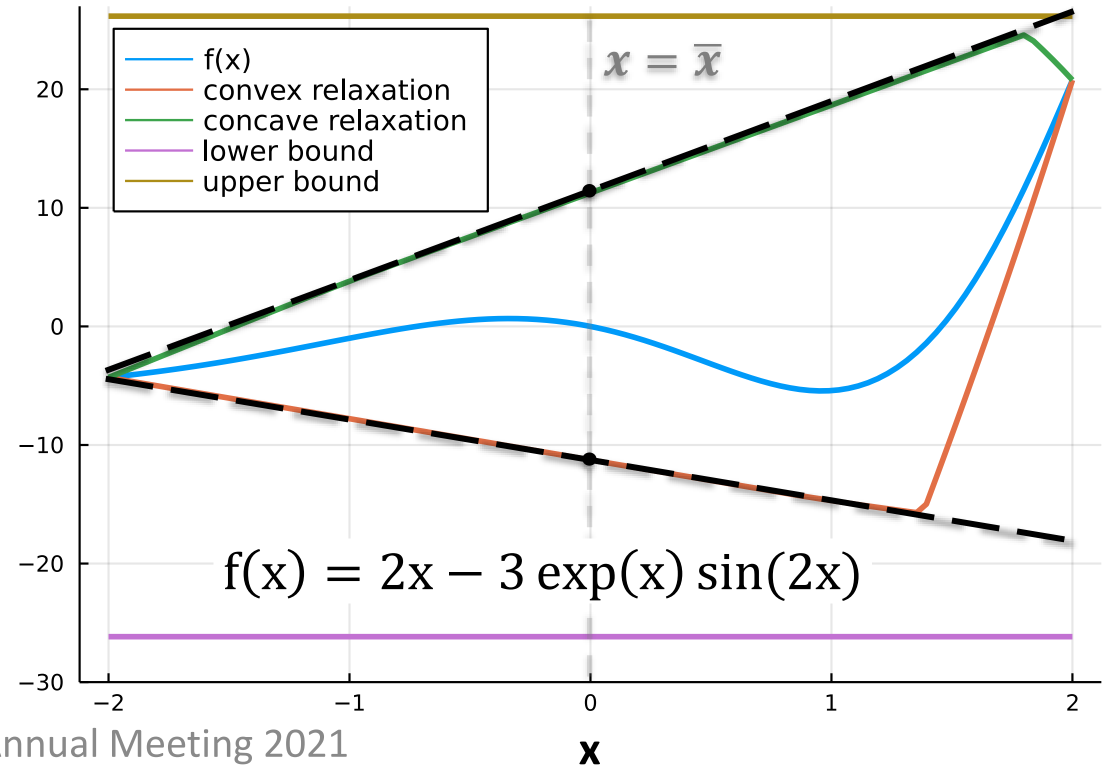
Use **affine relaxations** implied by **subgradients** computed using McCormick framework[4]

$$x(\mathbf{z}) \geq x^{cv}(\bar{\mathbf{z}}) + \nabla s_x^{cv}(\bar{\mathbf{z}})(\mathbf{z} - \bar{\mathbf{z}})$$

$$x(\mathbf{z}) \leq x^{cc}(\bar{\mathbf{z}}) + \nabla s_x^{cc}(\bar{\mathbf{z}})(\mathbf{z} - \bar{\mathbf{z}})$$

$$y(\mathbf{z}) \geq y^{cv}(\bar{\mathbf{z}}) + \nabla s_y^{cv}(\bar{\mathbf{z}})(\mathbf{z} - \bar{\mathbf{z}})$$

$$y(\mathbf{z}) \leq y^{cc}(\bar{\mathbf{z}}) + \nabla s_y^{cc}(\bar{\mathbf{z}})(\mathbf{z} - \bar{\mathbf{z}})$$



4. Mitsos, A, et al. **McCormick-based relaxations of algorithms.** *SIAM Journal on Optimization*, 20, (2009): 573 - 601.

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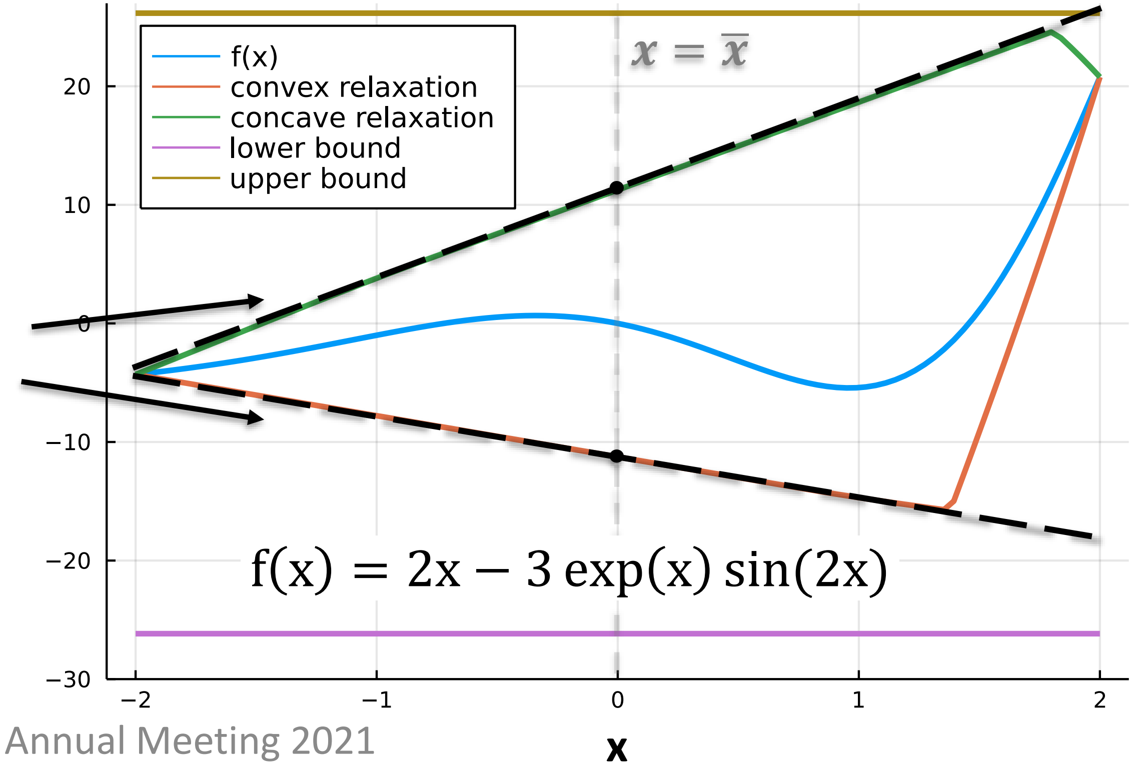
$$x(\mathbf{z}) \geq x^{cv}(\bar{\mathbf{z}}) + \nabla s_x^{cv}(\bar{\mathbf{z}})(\mathbf{z}-\bar{\mathbf{z}}) = u_x(\mathbf{z}) = u_x^{cv}(\mathbf{z})$$

$$x(\mathbf{z}) \leq x^{cc}(\bar{\mathbf{z}}) + \nabla s_x^{cc}(\bar{\mathbf{z}})(\mathbf{z}-\bar{\mathbf{z}}) = u_x(\mathbf{z}) = u_x^{cc}(\mathbf{z})$$

$$y(\mathbf{z}) \geq y^{cv}(\bar{\mathbf{z}}) + \nabla s_y^{cv}(\bar{\mathbf{z}})(\mathbf{z}-\bar{\mathbf{z}}) = u_y(\mathbf{z}) = u_y^{cv}(\mathbf{z})$$

$$y(\mathbf{z}) \leq y^{cc}(\bar{\mathbf{z}}) + \nabla s_y^{cc}(\bar{\mathbf{z}})(\mathbf{z}-\bar{\mathbf{z}}) = u_y(\mathbf{z}) = u_y^{cc}(\mathbf{z})$$



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Approach 2: Subgradients

Where to get convex/concave relaxations of a priori estimators (and extremal values)?

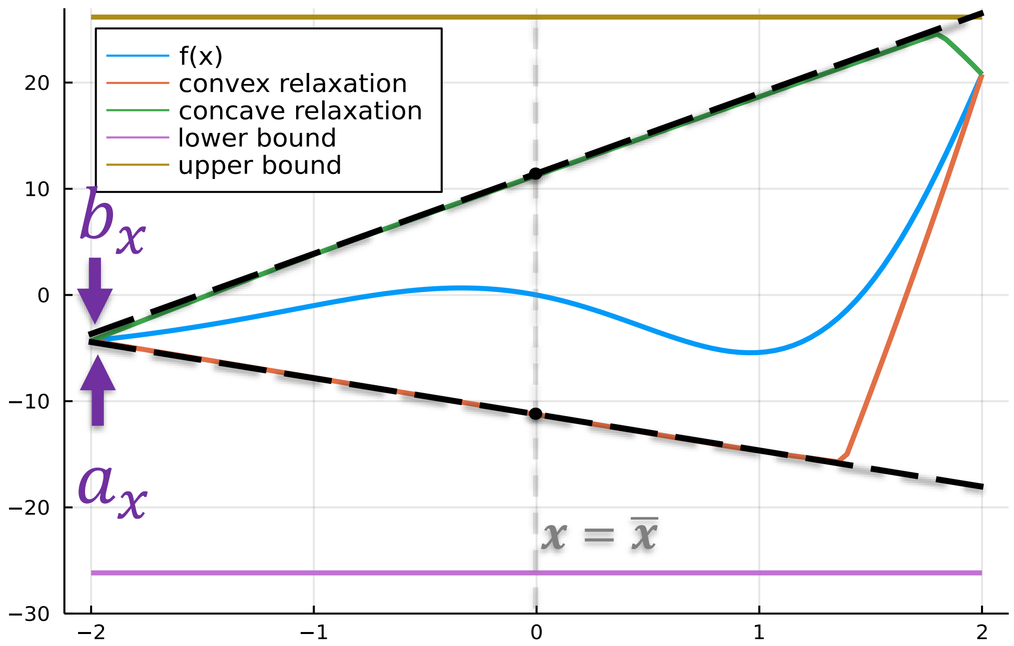
Use **affine relaxations** implied by **subgradients** computed using McCormick framework[8,18-20]

$$a_x = \left(x^{cv}(\bar{\mathbf{z}}) + \nabla s_x^{cv}(\bar{\mathbf{z}})(Z - \bar{\mathbf{z}})\right)^U \geq u_x^{cv}(\mathbf{z})$$

$$b_x = \left(x^{cc}(\bar{\mathbf{z}}) + \nabla s_x^{cc}(\bar{\mathbf{z}})(Z - \bar{\mathbf{z}})\right)^L \leq u_x^{cc}(\mathbf{z})$$

$$a_y = \left(y^{cv}(\bar{\mathbf{z}}) + \nabla s_y^{cv}(\bar{\mathbf{z}})(Z - \bar{\mathbf{z}})\right)^U \geq u_y^{cv}(\mathbf{z})$$

$$b_y = \left(y^{cc}(\bar{\mathbf{z}}) + \nabla s_y^{cc}(\bar{\mathbf{z}})(Z - \bar{\mathbf{z}})\right)^L \leq u_y^{cc}(\mathbf{z})$$



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8. Stuber, Matthew D., Joseph K. Scott, and Paul I. Barton. **Convex and concave relaxations of implicit functions.** *Optimization Methods and Software* 30.3 (2015): 424-460.
9. Najman, J., Mitsos, A. **Tighter McCormick relaxations through subgradient propagation.** *Journal of Global Optimization* **75,** 565–593 (2019).

1-D Example

Function

$$f(z) = (z - z^2)(z^3 - \exp(z)), \quad z \in [-0.5, 1]$$

- **Subgradient-Prior:** Use plane defined by subgradient at reference point for under/over-estimator.
- **Max-Concave:** McCormick relaxation are used as under/over-estimators.
- **Interval:** Use interval extensions of subgradients to tighten natural interval bounds for each factor.

  - McCormick Relaxation[8]
  - Multivariate McCormick Relaxation[11]
  - A priori Calculation (Subgradient)
  - A priori Calculation (Max-Concave)

8. Mitsos, A, et al. **McCormick-based relaxations of algorithms.** *SIAM Journal on Optimization*, 20, (2009): 573 - 601.
11. Tsoukalas, A., and Mitsos, A. **Multivariate McCormick relaxations.** *Journal of Global Optimization* 59.2-3 (2014): 633-662.

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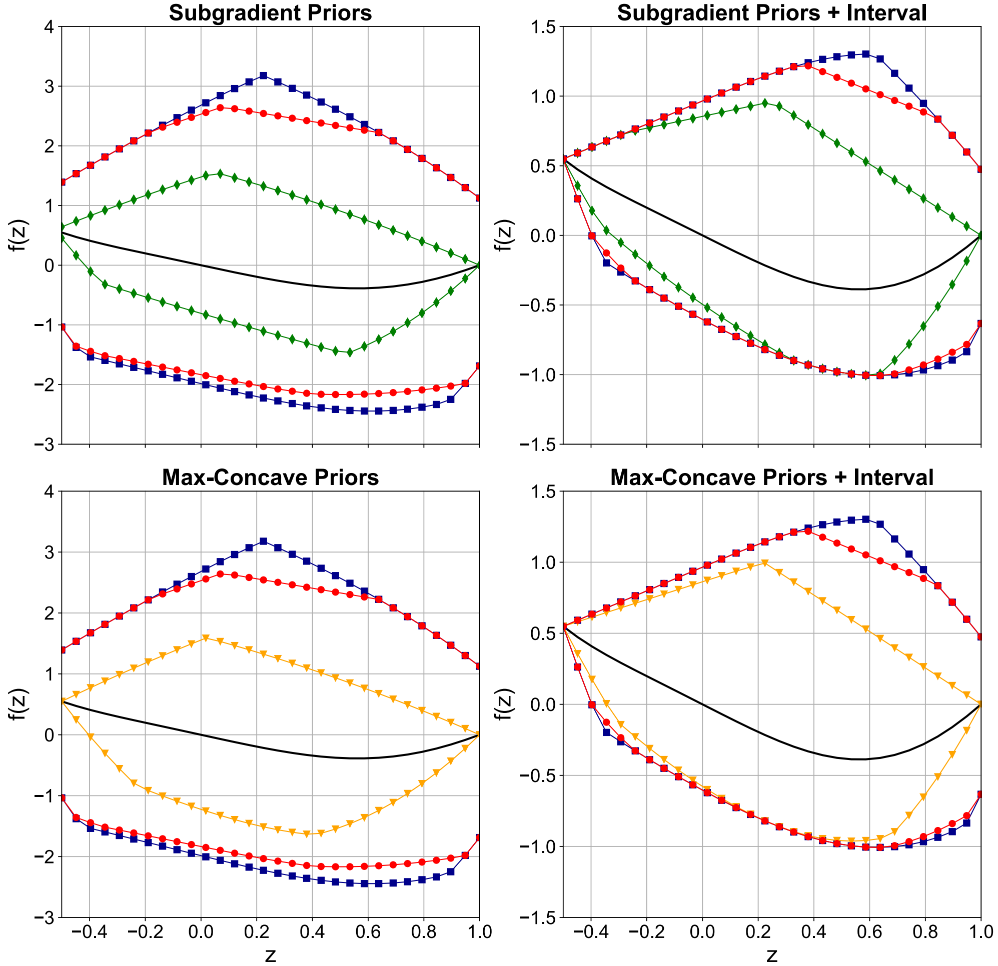

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1-D Example

Function

$f(z) =$

❖ **Subg**... ient at re...

❖ **Max**... unde...

❖ **Inter**... to tight...

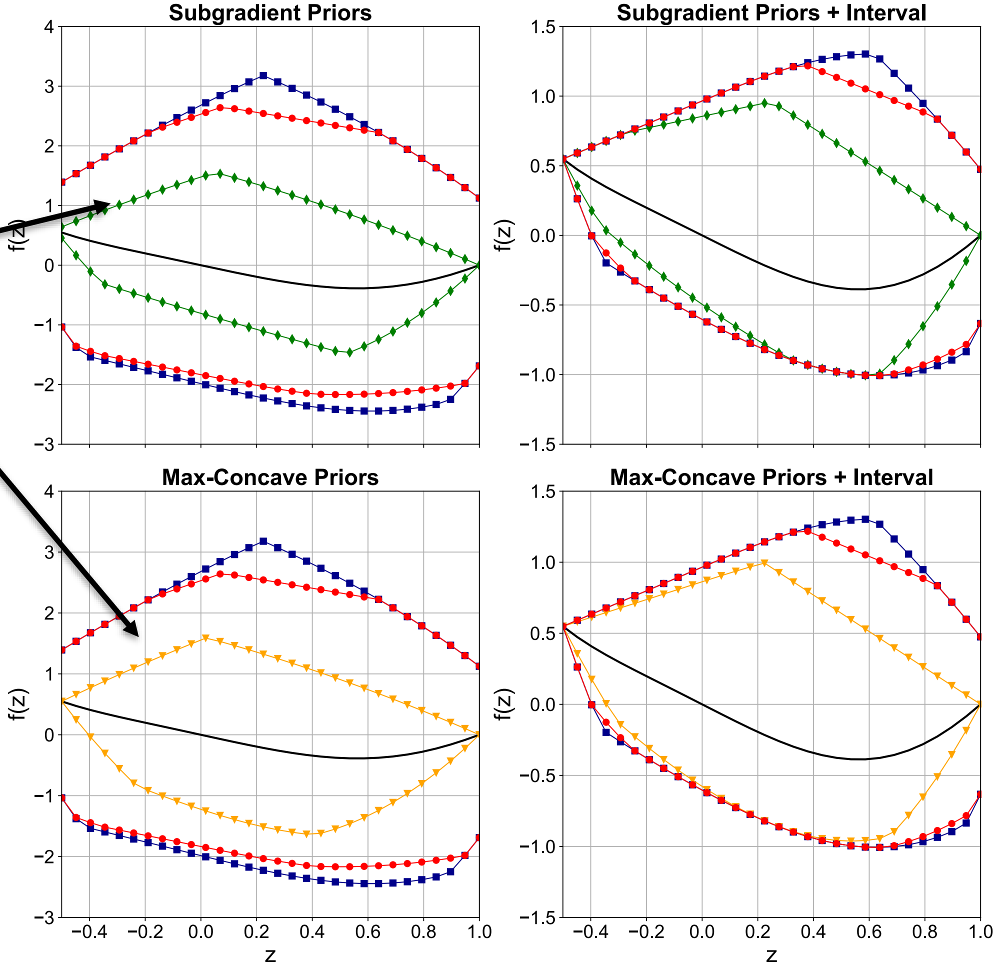
Observation

Subgradient-based a priori derived from McCormick relaxations are tighter than directly use of McCormick relaxation

Rationale

$$w \geq \underbrace{(y^U - y^L)u_x}_{} + y^L x + \underbrace{a_x(y - y^U)}_{}$$

Improving u_x may increase or decrease a_x



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11. Tsoukalas, A., and Mitsos, A. **Multivariate McCormick relaxations.** *Journal of Global Optimization* 59.2-3 (2014): 633-662.

Approach 3: Affine Arithmetic

AF1 – Affine Form[16]:

$$v = v_0 + \sum_{i=1}^{n} v_i \epsilon_i + v_{n+1}\epsilon_{\pm}$$

where $\epsilon_i, \epsilon_{\pm} \in [-1,1]$

$$v(\mathbf{z}) \geq v_0 + \sum_{i=1}^{n} v_i \epsilon_i(\mathbf{z}) - v_{n+1}$$

$$v(\mathbf{z}) \leq v_0 + \sum_{i=1}^{n} v_i \epsilon_i(\mathbf{z}) + v_{n+1}$$

$$\epsilon_i(\mathbf{z}) = \frac{z_i - mid(Z_i)}{rad(Z_i)}$$

AF2 – Affine Form[16]:

$$v = v_0 + \sum_{i=1}^{n} v_i \epsilon_i + v_{n+1}\epsilon_{\pm} + v_{n+2}\epsilon_{+} + v_{n+3}\epsilon_{-}$$

where $\epsilon_i, \epsilon_{\pm} \in [-1,1]$, $\epsilon_{+} \in [0,1]$, $\epsilon_{-} \in [-1,0]$

$$v(\mathbf{z}) \geq v_0 + \sum_{i=1}^{n} v_i \epsilon_i(\mathbf{z}) - v_{n+1} - v_{n+3}$$

$$v(\mathbf{z}) \leq v_0 + \sum_{i=1}^{n} v_i \epsilon_i(\mathbf{z}) + v_{n+1} + v_{n+2}$$

16. Messine, F.: **Extensions of affine arithmetic: Application to unconstrained global optimization.** *JUCS* (11), 992–1015 (2002).
17. Ninin, J., Messine, F., Hansen, P.: **A reliable affine relaxation method for global optimization.** *4OR13* (3), 247–277 (2015).

Approach 3: Affine Arithmetic

$$w = \underbrace{(x^2 - x)}_{z_1} \times \underbrace{(y^2 - y)}_{z_2},$$

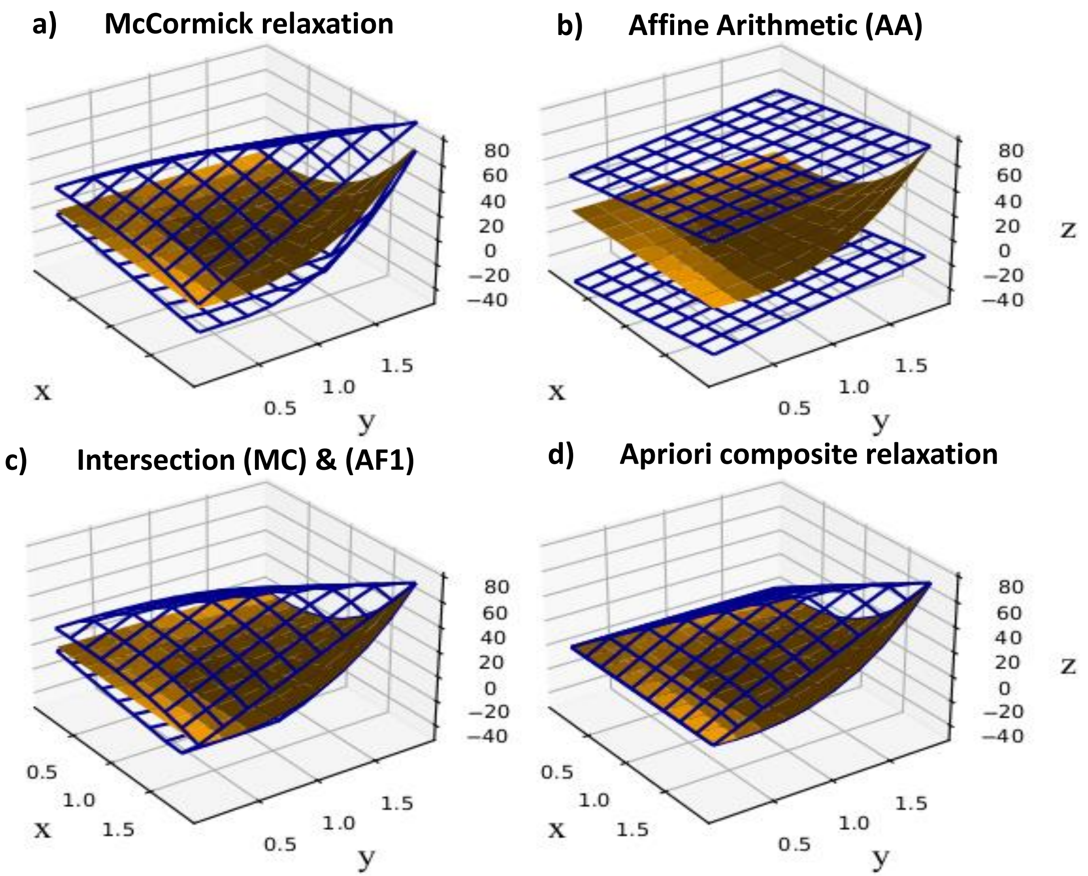
$$X \times Y = [0.1, 0.9]^2$$

a) McCormick relaxation (MC)[8]

b) Affine relaxations from affine arithmetic (AF1)[22,23]

c) Intersection of MC & AF1 on each factor

d) A priori composite relaxations (via AA)



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Implementation

- IntervalArithmetic.jl[24] used for validated calculations.
- Relaxations computed via McCormick.jl[25] (v0.11.1)
- EAGO.jl[26] (v0.7.0) global optimizer used to solve problems in Julia[27] 1.6.2. Tolerances set to 1E-4.
- In-house code implementation of AF2 affine arithmetic using IntervalArithmetic.jl for validated calculations.

https://www.github.com/PSORLab/EAGO.jl

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Benchmarking Problems

Adapted from Example 5[16]

$$\min_{\mathbf{x} \in X} \langle \mathbf{c}, \mathbf{x} \rangle + \sum_{i \geq j} q_{i,j} y_i(\mathbf{x}) y_j(\mathbf{x})$$

$$\mathbf{y}(\mathbf{x}) = (x_1^2, x_1^3, x_1^4, \cdots, x_n^2, x_n^3, x_n^4)$$

$$\mathbf{x} \in X = [-2,2]^n$$

25x instances for $n \in \{5,7,9\}$ and v proportion of nonzero entries of Q with $\mathrm{v} \in \{0.3, 0.5, 0.7\}$ were generated for randomly such that

$$q_{i,j} \sim U(-2, 2) \qquad c_i \sim U(-512, -2)$$

Optimizer	CPU Time (s)	% Solved (<100s)
SCIP	9.87s	96.4%
EAGO	7.89s	96.8%
EAGO + A priori (Subgradient)	**5.42s**	**99.5%**
EAGO + A priori (Affine)	7.42	98.2%

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Dynamic Applications

Objective $\min_{\mathbf{p} \in P} \sum_{1 \le i \le 3} \mathrm{x}_i(\mathbf{p}, t_f)$

$0 \le t \le t_f = 0.1$

$\mathbf{p} \in P = [0,30]^3$

Initial Condition $\mathbf{x}_0(\mathbf{p}) = \mathbf{p}$

ODE System

$$\mathrm{x}_1'(\mathbf{p}, t) = -\mathrm{x}_1(\mathbf{p}, t)\mathrm{x}_2(\mathbf{p}, t) + \mathrm{x}_3(\mathbf{p}, t)$$

$$\mathrm{x}_2'(\mathbf{p}, t) = 2\mathrm{x}_1(\mathbf{p}, t)\mathrm{x}_2(\mathbf{p}, t)$$

$$\mathrm{x}_3'(\mathbf{p}, t) = \mathrm{x}_1(\mathbf{p}, t) + \mathrm{x}_2(\mathbf{p}, t)$$

Method	CPU Time (s)	Iterations
McCormick relaxation	22.1	692k
A priori relaxation (via subgradient)	**12.7**	**134k**

Euler Discretization (h = 0.01)

$$\mathrm{x}_{i+1,1} = \mathrm{x}_{i,1} + h\left(-\mathrm{x}_{i,1}\mathrm{x}_{i,2} + \mathrm{x}_{i,3}\right)$$

$$\mathrm{x}_{i+1,2} = \mathrm{x}_{i,2} + h(2\mathrm{x}_{i,1}\mathrm{x}_{i,2})$$

$$\mathrm{x}_{i+1,3} = \mathrm{x}_{i,3} + h\left(\mathrm{x}_{i,1} + \mathrm{x}_{i,2}\right)$$

Future Work

- ❑ Investigate a priori relaxations for min, max, mid, and division operators
- ❑ Extension to relaxation of multilinear operators such as trilinear envelopes[21,22]
- ❑ Evaluate effect on existing algorithms:
  - Relaxation of parametric ordinary differential equations[23]
  - Convex/concave relaxations of implicit functions[24]
- ❑ Multivariate treatment of constrained McCormick relaxation

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