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Recent Advances in EAGO.jl: A Feature-Rich Global Solver and Research Platform

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informs ANNUAL MEETING
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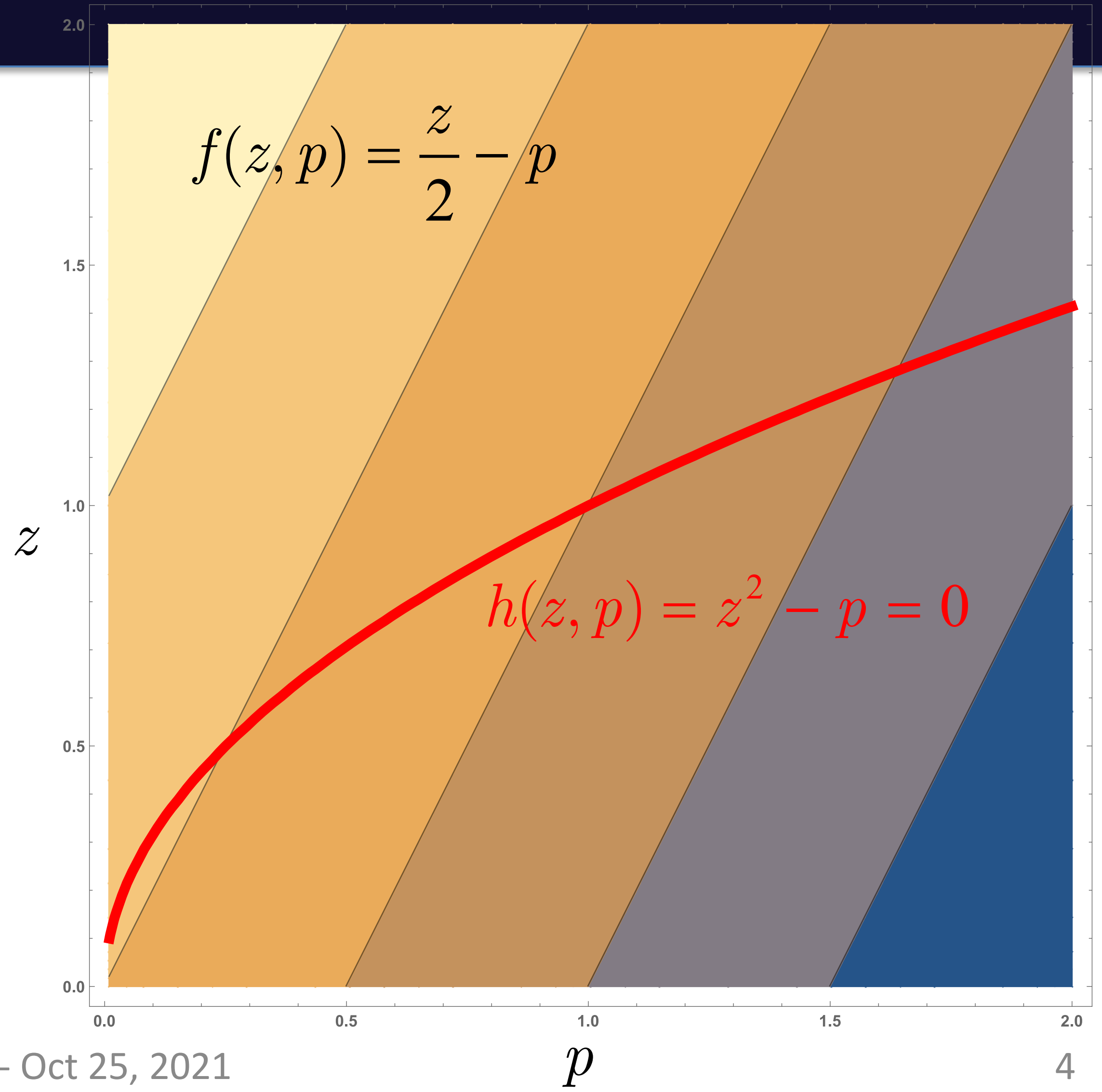
EAGO.jl developer

Outline

- Motivation
  - Reduced-space deterministic global optimization
- EAGO.jl: Deterministic global optimization in Julia
  - Core features
  - Main features for advanced formulations
  - New and near-future additions
- Conclusions

Motivation: Reduced-Space Optimization

$$\min_{z,p} \frac{z}{2} - p$$
$$\text{s.t.} \quad z^2 - p = 0$$
$$z \in [0,2]$$
$$p \in [0.01,2]$$

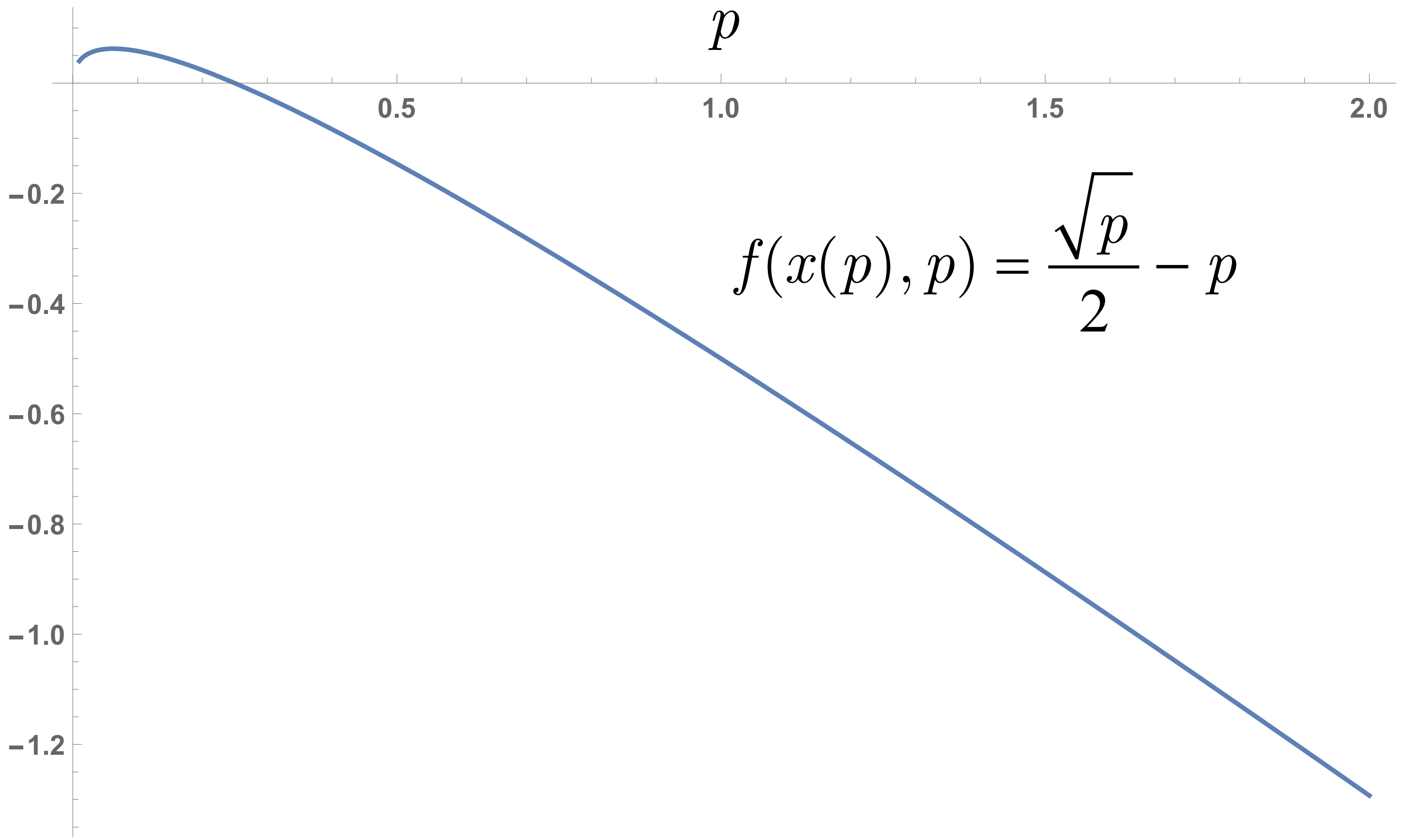


Motivation: Reduced-Space Optimization

$$z^2 - p = 0$$

$$\Rightarrow x(p) = \sqrt{p}$$

$$\min_p \frac{x(p)}{2} - p$$
$$\text{s.t. } p \in [0.01, 2]$$



Motivation: Reduced-Space Optimization

Want to solve dynamic optimization problems to guaranteed global optimality:

$$\phi^* = \min_{\mathbf{p}\in P\subset\mathbb{R}^{n_p}} \phi(\mathbf{x}(\mathbf{p},t_f),\mathbf{p})$$

$$\text{s.t. } \dot{\mathbf{x}}(\mathbf{p},t) = \mathbf{f}(\mathbf{x}(\mathbf{p},t),\mathbf{p},t),\ \forall t \in I = [t_0,t_f]$$

$$\mathbf{x}(\mathbf{p},t_0) = \mathbf{x}_0(\mathbf{p})$$

$$\mathbf{g}(\mathbf{x}(\mathbf{p},t_f),\mathbf{p}) \le \mathbf{0}$$

Dimensionality: n_p

$$\phi^* = \min_{\mathbf{p}\in P,\hat{\mathbf{z}}\in Z} \phi(\hat{\mathbf{z}},\mathbf{p},t_f)$$

$$\text{s.t. } \mathbf{z}_0 = \mathbf{x}_0(\mathbf{u},\mathbf{p})$$

$$\hat{\mathbf{z}}_1 - \mathbf{z}_0 - h\mathbf{f}(\hat{\mathbf{z}}_1,\mathbf{p},t_1) = \mathbf{0}$$

$$\vdots \qquad \vdots$$

$$\hat{\mathbf{z}}_K - \hat{\mathbf{z}}_{K-1} - h\mathbf{f}(\hat{\mathbf{z}}_K,\mathbf{p},t_K) = \mathbf{0}$$

$$\mathbf{g}(\hat{\mathbf{z}}_K,\mathbf{p}) \le \mathbf{0}$$

Dimensionality: $n_p \times K$

Background: EAGO

How do you get EAGO?

From Julia package manager:

```
(@v1.6) pkg> add EAGO
```

```
julia> using Pkg;

julia> Pkg.add("EAGO")
```

From GitHub:

https://www.github.com/PSORLab/EAGO.jl

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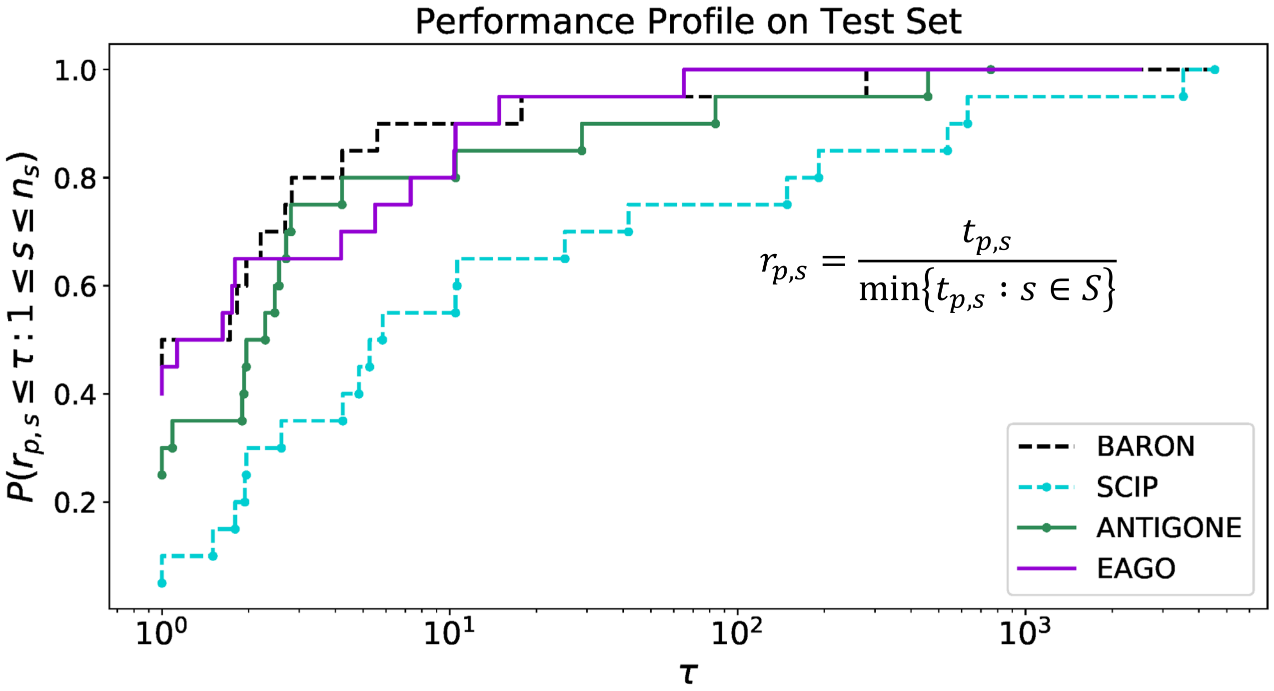
How do you use EAGO?

As a solver in the open-source algebraic modeling language JuMP.

As a stand-alone solver.

Published Results

- EAGO exhibits competitive performance on benchmarking set



OPTIMIZATION METHODS & SOFTWARE
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EAGO.jl: easy advanced global optimization in Julia

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ABSTRACT

An extensible open-source deterministic global optimizer (EAGO) programmed entirely in the Julia language is presented. EAGO was developed to serve the need for supporting higher-complexity user-defined functions (e.g. functions defined implicitly via algorithms) within optimization models. EAGO embeds a first-of-its-kind implementation of McCormick arithmetic in an Evaluator structure allowing for the construction of convex/concave relaxations using a combination of source code transformation, multiple dispatch, and context-specific approaches. Utilities are included to parse user-defined functions into a directed acyclic graph representation and perform symbolic transformations enabling dramatically improved solution speed. EAGO is compatible with a wide variety of local optimizers, the most exhaustive library of transcendental functions, and allows for easy accessibility through the JuMP modelling language. Together with Julia's minimalist syntax and competitive speed, these powerful features make EAGO a versatile research platform enabling easy construction of novel meta-solvers, incorporation and utilization of new relaxations, and extension to advanced problem formulations encountered in engineering and operations research (e.g. multilevel problems, user-defined functions). The applicability and flexibility of this novel software is demonstrated on a diverse set of examples. Lastly, EAGO is demonstrated to perform comparably to state-of-the-art commercial optimizers on a benchmarking test set.

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Deterministic global optimization; nonconvex programming; McCormick relaxations; optimization software; branch-and-bound; Julia

2010 MATHEMATICS SUBJECT CLASSIFICATIONS
90C26; 90C34; 90C57; 90C90

1. Introduction and motivation

Mathematical optimization problems are ubiquitous in scientific and technical fields. Applications range from aerospace and chemical process systems to finance. However, even relatively simple physical processes such as mixing, may introduce significant nonconvexity into problem formulations [60]. As such, nonconvex programs often represent the most faithful representations of the system of interest. Multiple approaches have been developed to address these cases. Heuristics such as evolutionary algorithms, may approximate good solutions for select problems. However, heuristics may fail to guarantee that even a feasible

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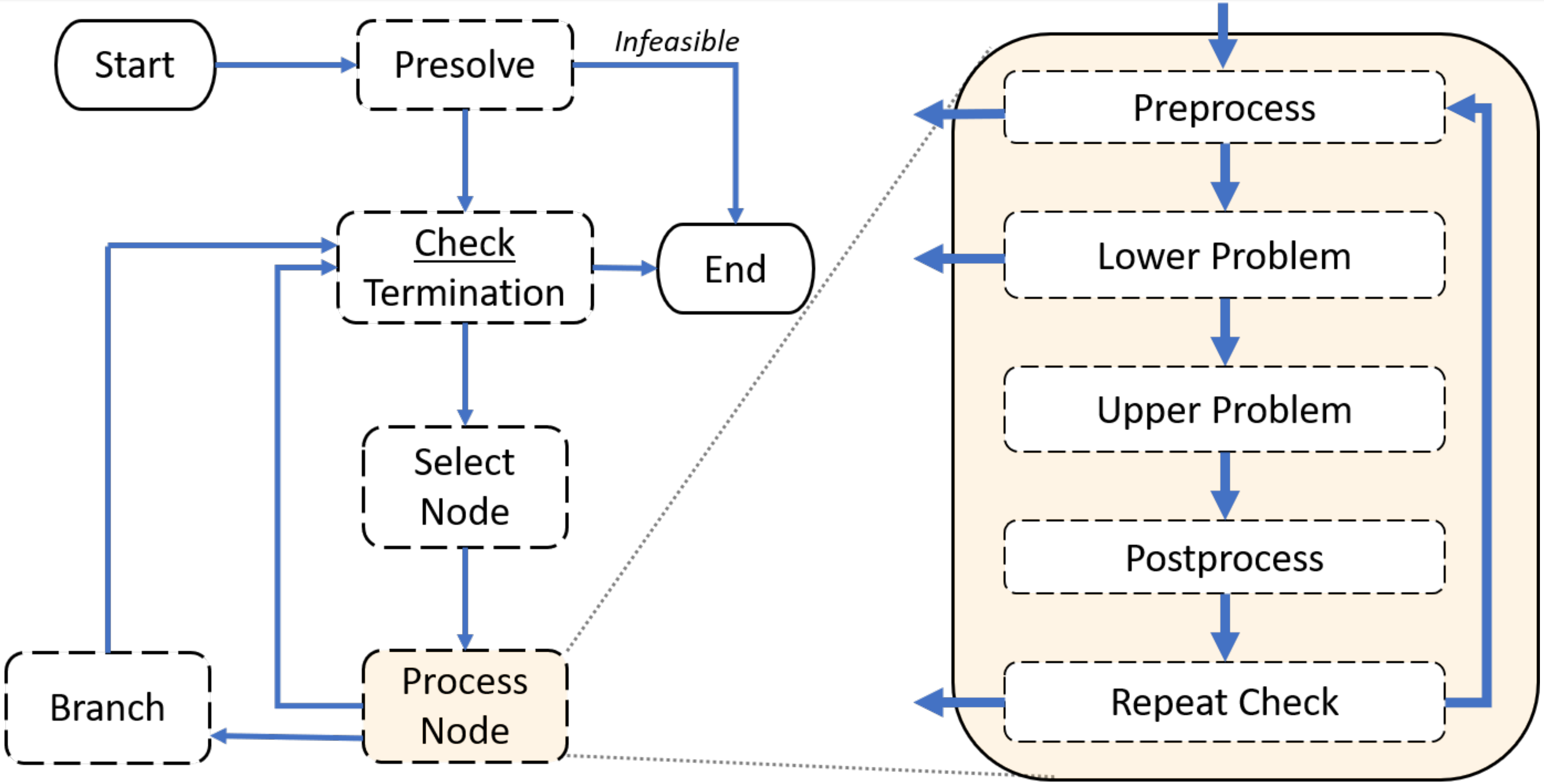
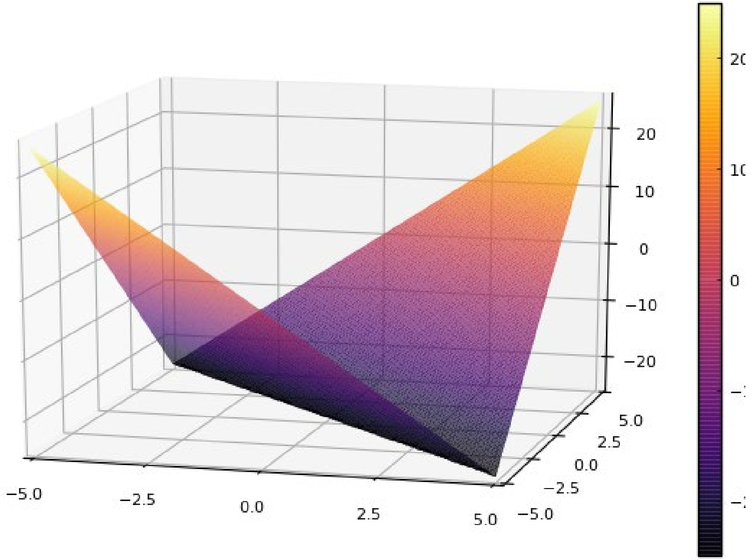
EAGO.jl: Core Optimizer



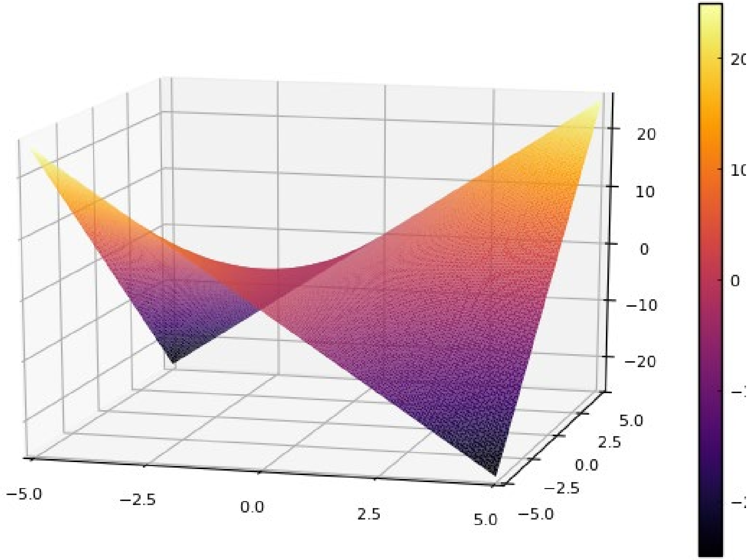
McCormick-Based Relaxations

- Most broadly known for convex/concave relaxations of bilinear terms

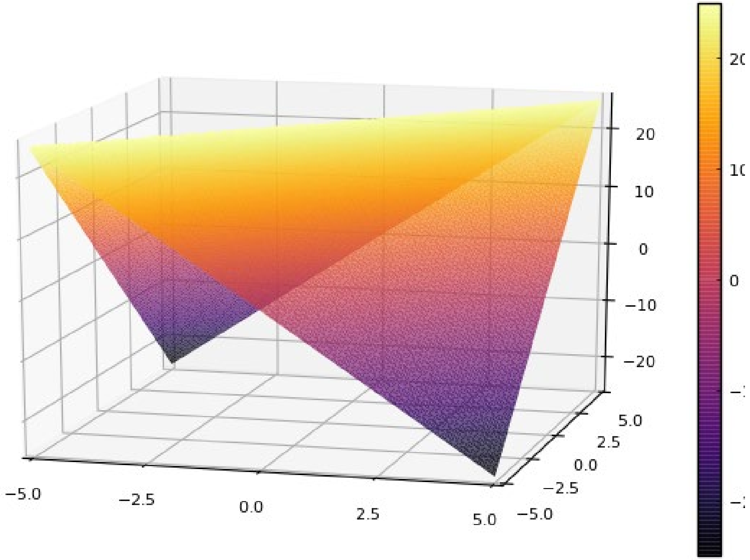
$f^{cv}(x, y)$



$f(x, y) = xy$



$f^{cc}(x, y)$



Convex/Concave Relaxations

$$f(x) = x^2(1 - \exp(-x))$$

Auxiliary Variable Method

$$y_1 = x^2$$
$$y_2 = 1 - \exp(-x)$$
$$y_3 = y_1 y_2$$

$$(x^2)^{cv} \le y_1 \le (x^2)^{cc}$$
$$(1 - \exp(-x))^{cv} \le y_2 \le (1 - \exp(-x))^{cc}$$
$$(y_1 y_2)^{cv} \le y_3 \le (y_1 y_2)^{cc}$$

An optimization formulation is “lifted” from 1 original decision variable to 4.

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$$(y_1 y_2)^{cv} \le y_3 \le (y_1 y_2)^{cc}$$

An optimization formulation is "lifted" from 1 original decision variable to 4.

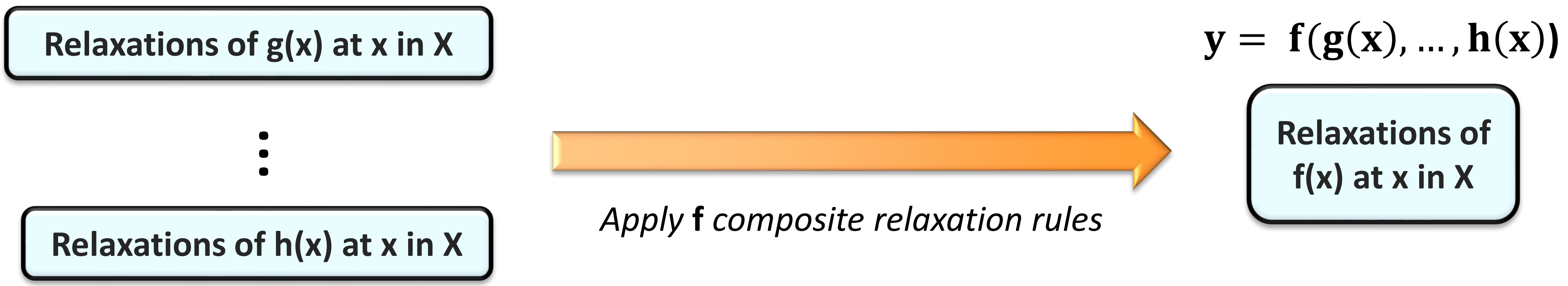
McCormick

$$y_1(x) = x^2$$
$$y_2(x) = 1 - \exp(-x)$$
$$y_3(x) = y_1(x) y_2(x)$$
$$f(x) = y_3(x)$$

$$f^{cv}(x) \le f(x) \le f^{cv}(x)$$

An optimization formulation with 1 original decision variable remains in the original dimensionality space.

EAGO.jl: McCormick Relaxations

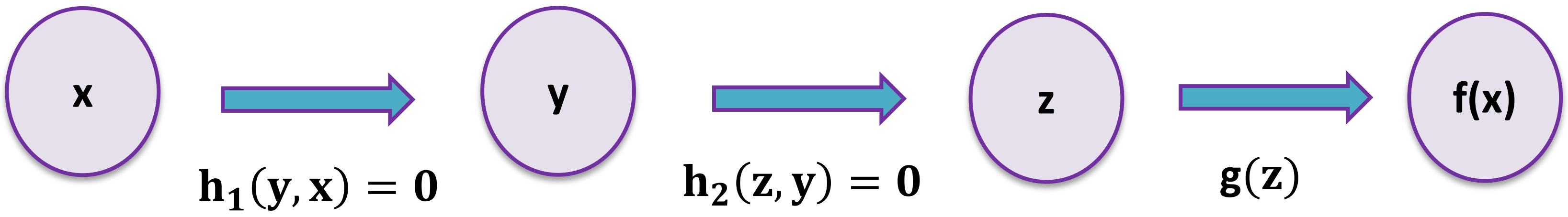


- Improved (tighter) relaxations of composite bilinear and trilinear terms*
- Supports a variety of nonlinear expressions:
  - **Common algebraic expressions:** *log, log2, log10, exp, exp2, exp10, sqrt, +, -, ^, min, max, /,* x, *abs, step, cbrt, …*
  - **Trigonometric Functions:** *sin, cos, tan, asin, acos, atan, sec, csc, cot, asec, acsc, acot…*
  - **Hyperbolic Functions:** *sinh, cosh, tanh, asinh, acosh, atanh, sech, csch, coth, acsch, acoth*
  - **Special Functions:** *erf, erfc, erfinv, erfcinv*
  - **Activation Functions**:** *relu, leaky_relu, sigmoid, softsign, softplus, maxtanh, gelu, elu, selu, silu, …*
  - ***Common Algebraic Expressions:*** *xlogx, arh, xexpax*

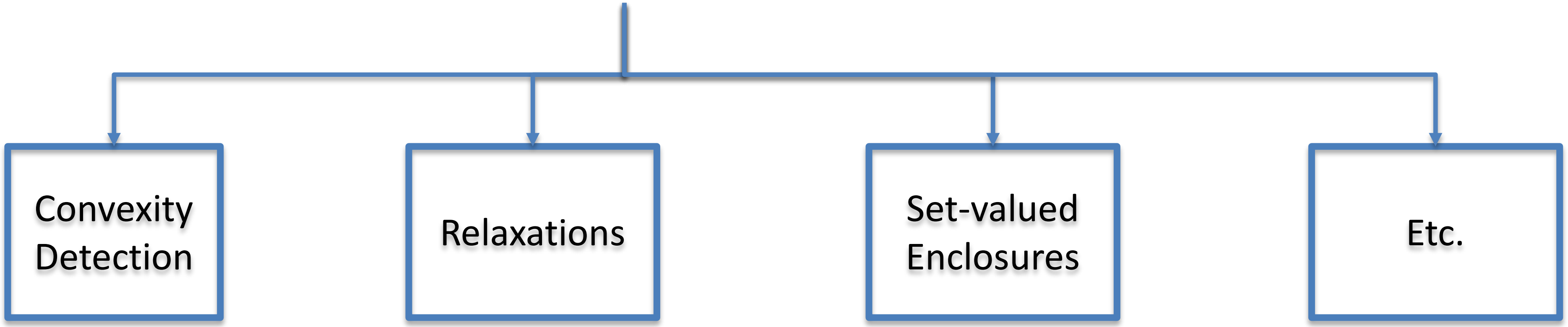
In preparation* * Under Review*

EAGO.jl: New Multigraph Backend

Introduce support for multiple-output subexpressions.



Separate caches of information from graph structure (extendibility).



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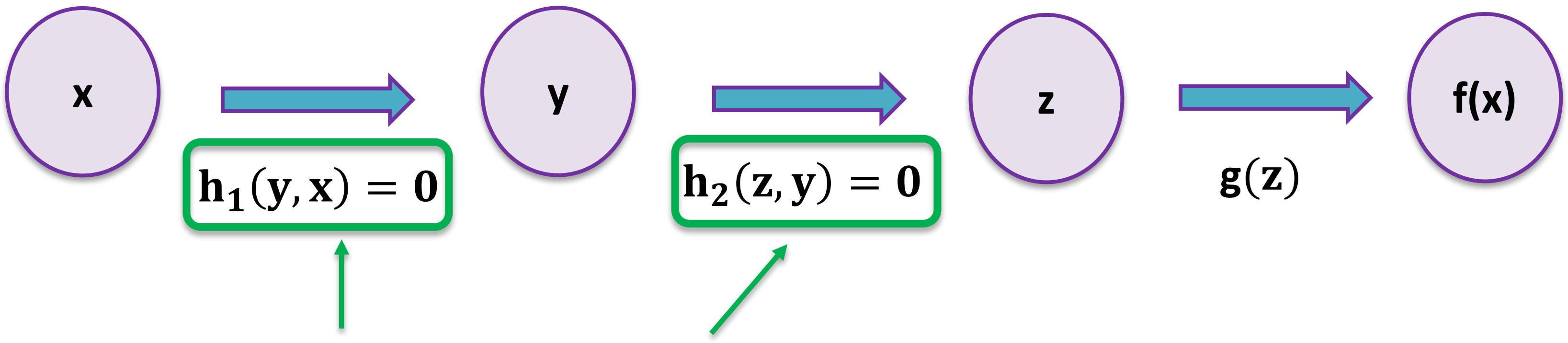


- Introduce support for **multiple-output subexpressions** into graph representation (e.g., h_1, h_2).
- Support introduction of auxiliary variables (distinct from decision variables).
- Allows for chaining of computation of auxiliary variables and general implicit functions.

EAGO.jl: Core Optimizer

Key Improvements to Global Optimization Routine:

- Heuristics to ensure numerically safe affine relaxations for lower-bounding problems
- More computationally efficient approach to optimization-based bounds tightening
- No-overhead user-defined subroutines (lower-bounding problem, etc.)
- Improved parameter tuning

Other High-Level Improvements to EAGO's Global Optimizer:

- Bridging + configuration for a large variety of subsolvers
- Preliminary support for integer-variables (MINLP problem forms)
- Detection of specialized problem forms (LP, MILP, convex)
- Support for additional semi-infinite programming (SIP) routines

Improvements to MINLP solution algorithm currently under development.

EAGO.jl: Novel Relaxations

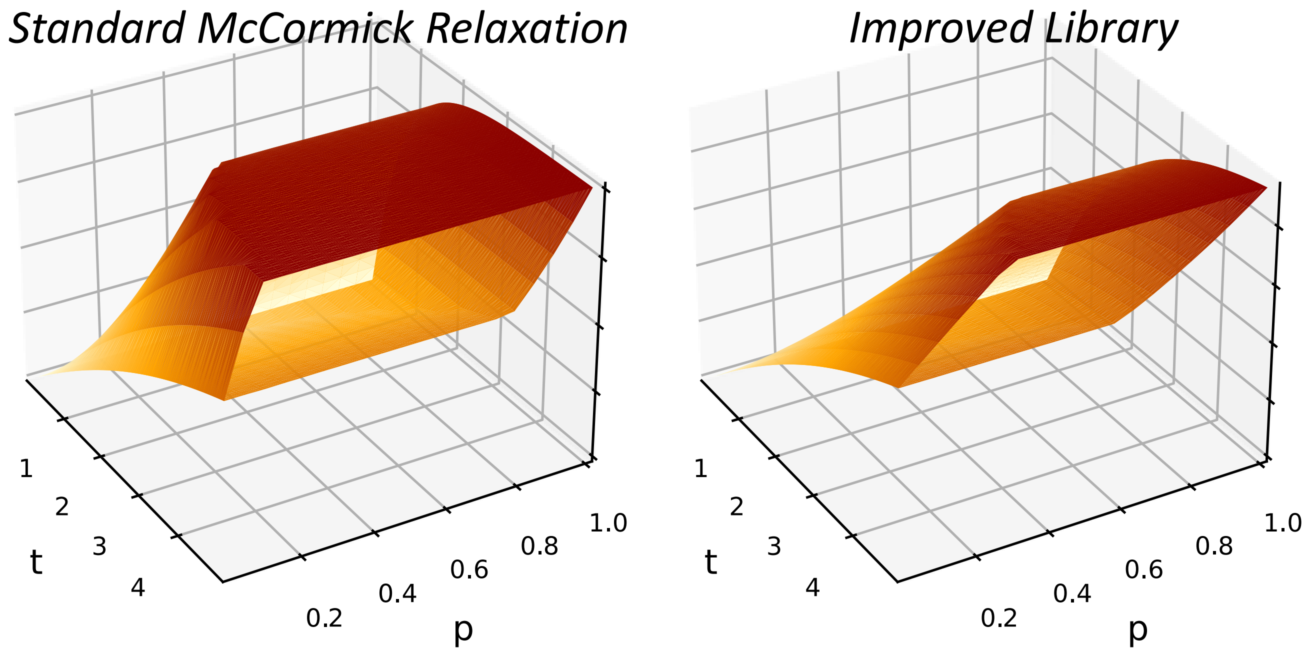
Improved relaxation subroutine performance due to intrinsic relaxation library upgrade:

- Relaxation of implicit functions[10]
- Relaxations of ODEs[11]
- Reverse propagation of relaxations[12]

Simple ODE Relaxation

$$\frac{dx}{dt} = \exp(p)\sin(x)(2 - x),$$

$$x(0) = 1, \quad p \in [0.01,1], \quad t \in [0,5]$$



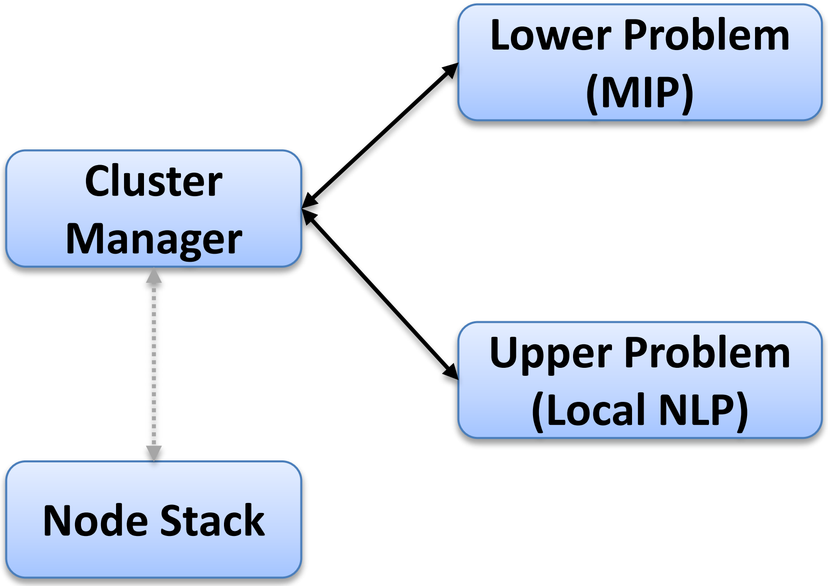
10. Stuber, MD et al. **Convex and concave relaxations of implicit functions.** *Optimization Methods and Software* (2015), 30, 424-460
11. Scott, Joseph K., and Paul I. Barton. **Improved relaxations for the parametric solutions of ODEs using differential inequalities.** *Journal of Global Optimization* 57.1 (2013): 143-176.
12. Wechsung, Achim, et al. **Reverse propagation of McCormick relaxations.** *Journal of Global Optimization* 63.1 (2015): 1-36.

EAGO.jl: Distributed Computing

Overall Framework:

Currently, under development. Expected by end of 2021.

- Support for distributed computing via ClusterManager abstraction
- User-specified entry point for parallelism
  - Parallel evaluation of relaxations
  - Parameter setting in optimizers used by subproblem
  - Lower bounding, upper bounding subproblems



EAGO.jl: Main Features

- ***Embedded Machine Learning (ML) Models***

  Wilhelm and Stuber, VSD63 (Sunday, Virtual Room 63)
- Semi-infinite Programming
- Dynamic Optimization

… Composability thereof

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… Composability thereof

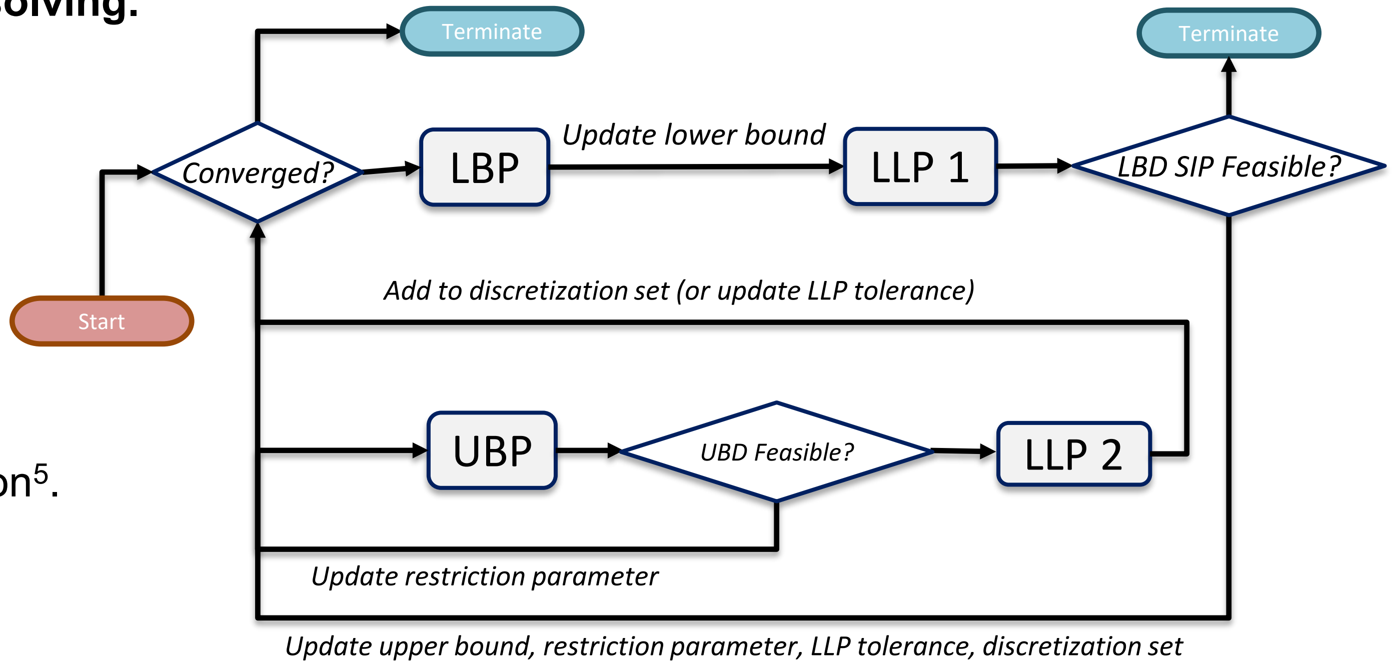
Solving Nonconvex SIPs

- **EAGO.jl supports for general nonconvex SIP solving.**

$$f^* = \min_{\mathbf{x}\in X} f(\mathbf{x})$$
$$\text{s.t.} \quad g(\mathbf{x},\mathbf{p}) \leq 0, \forall \mathbf{p} \in P$$

- **Composable with ML/dynamic relaxations.**
- **New features:**
  - Added new hybrid-oracle SIP routine[5].
  - Automatic subproblem tolerance specification[5].
  - User-extendable SIP subproblems.

SIPres algorithm[4]



3. B. Bhattacharjee, P. Lemonidis, W.H. Green Jr, and P.I. Barton. **Global solution of semi-infinite programs.** *Math. Program*. 103 (2005), pp. 283–307.
4. Mitsos, Alexander. **Global optimization of semi-infinite programs via restriction of the right-hand side.** *Optimization* 60.10-11 (2011): 1291-1308.
5. Djelassi, Hatim, and Alexander Mitsos. **A hybrid discretization algorithm with guaranteed feasibility for the global solution of semi-infinite programs.** *Journal of Global Optimization* 68.2 (2017): 227-253.

EAGO.jl: Main Features

- Embedded Machine Learning (ML) Models
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- Semi-infinite Programming
- ***Dynamic Optimization***

… Composability thereof

Dynamics, Relaxation

- New support included for general nonlinear parametric ordinary differential equations.
- Incorporation into global optimizer:
  - Relaxations & Domain Reduction:
    - Interval bounds
    - Relaxations & (sub)gradients
  - Local NLP solver:
    - Automatic differentiation for upper-bounding problem

Continuous-Time Relaxations[10,11,12]

$$\dot{\mathbf{x}}^{cv}(t,\mathbf{p}) = \mathbf{f}^{cv}(t,\mathbf{p},\mathbf{x}^{cv}(t,\mathbf{p}),\mathbf{x}^{cc}(t,\mathbf{p})), \quad \mathbf{x}^{cv}(t_0,\mathbf{p}) = \mathbf{x}_0^{cv}(\mathbf{p})$$
$$\dot{\mathbf{x}}^{cc}(t,\mathbf{p}) = \mathbf{f}^{cc}(t,\mathbf{p},\mathbf{x}^{cv}(t,\mathbf{p}),\mathbf{x}^{cc}(t,\mathbf{p})), \quad \mathbf{x}^{cc}(t_0,\mathbf{p}) = \mathbf{x}_0^{cc}(\mathbf{p})$$

Discrete-Time Relaxations[13,14,15]

$$\mathbf{x}(\tau_{q+1},\mathbf{p}) \in \underbrace{\mathbf{x}(\tau_q,\mathbf{p}) + \sum_{j=1}^{p}\frac{h^j}{j!}\mathbf{f}^{(j)}(\mathbf{x}(\tau_q,\mathbf{p}),\mathbf{p})}_{\text{Taylor Series}} + \underbrace{\frac{h^{p+1}}{(p+1)!}\mathbf{f}^{(p+1)}(\mathrm{X}(\tau_q),\mathrm{P})}_{\text{Remainder Bound}}$$

10. Scott, Joseph K., and Paul I. Barton. **Improved relaxations for the parametric solutions of ODEs using differential inequalities.** *Journal of Global Optimization* 57.1 (2013): 143-176.
11. Scott, Joseph K., and Paul I. Barton. **Bounds on the reachable sets of nonlinear control systems.** *Automatica* 49.1 (2013): 93-100.
12. Scott, Joseph K., Benoit Chachuat, and Paul I. Barton. **Nonlinear convex and concave relaxations for the solutions of parametric ODEs.** *Optimal Control Applications and Methods* 34.2 (2013): 145-163.
13. Sahlodin, Ali M., and Benoit Chachuat. **Discretize-then-relax approach for convex/concave relaxations of the solutions of parametric ODEs.** *Applied Numerical Mathematics* 61.7 (2011): 803-820.
14. Berz, Martin, and Georg Hoffstätter. **Computation and application of Taylor polynomials with interval remainder bounds.** *Reliable Computing* 4.1 (1998): 83-97.
15. Sahlodin, Ali Mohammad, and Benoit Chachuat. **Convex/concave relaxations of parametric ODEs using Taylor models.** *Computers & Chemical Engineering* 35.5 (2011): 844-857.

Dynamics, Implementation

Core Algorithms

- **DynamicBoundspODEsDiscrete.jl**
  – Discrete time approaches
- **DynamicBoundspODEsIneq.jl**
  -- Continuous time approaches

Abstraction Layer

DynamicBounds.jl[34]

- **DynamicBounds.jl**
- **DynamicBoundsBase.jl**

Extendable Global Optimizer[35]

EAGO

EAGODynamicOptimizer.jl[36]

Future Work: Integrate with JuMP-based frontends (e.g., InfiniteOpt.jl[37])

34. Wilhelm, M. E., **DynamicBounds.jl**, (2020), GitHub repository, https://github.com/PSORLab/DynamicBounds.jl
35. Wilhelm, M. E., and M. D. Stuber. **EAGO. jl: easy advanced global optimization in Julia.** *Optimization Methods and Software* (2020): 1-26.
36. Wilhelm, M. E., **EAGODynamicOptimizer.jl**, (2020), GitHub repository, https://github.com/PSORLab/EAGODynamicOptimizer.jl
37. Pulsipher, J.L., et al. A Unifying Modeling Abstraction for Infinite-Dimensional Optimization, https://arxiv.org/abs/2106.12689

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- Embedded Machine Learning (ML) Models
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- Semi-infinite Programming
- Dynamic Programming

*… **Composability thereof***

Robust Dynamic Optimization

Dynamic SIP Formulation

$$\Phi^* = \min_{\mathbf{u}} \Phi(\mathbf{u})$$ ← Objective

$$\text{s.t. } g(\mathbf{x}(\mathbf{u}, \mathbf{p}, \mathrm{t}_f), \mathbf{u}, \mathbf{p}) \leq 0$$ ← Performance Constraint(s)

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(\mathbf{u}, \mathbf{p}, \mathrm{t}), \mathbf{u}, \mathbf{p})$$ ← Parametric ODEs

$$\mathbf{x}(\mathbf{u}, \mathbf{p}, \mathrm{t}_0) = \mathbf{x}_0(\mathbf{u}, \mathbf{p})$$ ← Initial Condition

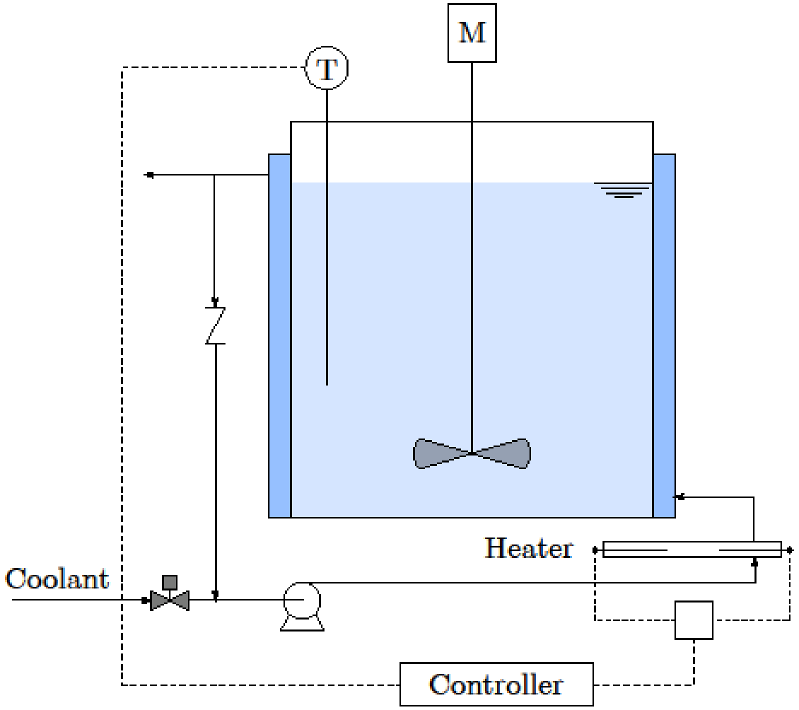
$$t \in I = [t_0, t_f], \forall \mathbf{p} \in P$$

- Design under worst-case realization of uncertainty.
- Safety-critical systems and high-impact defect elimination.

1. Puschke, Jennifer, et al. **Robust dynamic optimization of batch processes under parametric uncertainty: Utilizing approaches from semi-infinite programs.** *Computers & Chemical Engineering* 116 (2018): 253-267.
2. Puschke, Jennifer, and Alexander Mitsos. **Robust feasible control based on multi-stage eNMPC considering worst-case scenarios.** *Journal of Process Control* 69 (2018): 8-15.

Robust Dynamic Optimization

Batch MMA Polymerization Reaction



Adequate cooling at maximum temperature to withstand sensor fault?

Robust Operation SIP

$$\gamma^* = \max_{p \in P, \gamma \in \Gamma} \gamma$$
$$\text{s.t. } \gamma \leq T(t_f, \mathbf{u}) - p, \ \forall \mathbf{u} \in U$$

- ❑ Nonconvex semi-infinite program
- ❑ Embedded dynamic system
- ❑ Complex chemical kinetics (hybrid model desirable)

Robust Dynamic Optimization

Robust Operation SIP

$$\gamma^* = \max_{p \in P, \gamma \in \Gamma} \gamma$$
$$\text{s.t. } \gamma \leq T(t_f, \mathbf{u}) - p, \ \forall \mathbf{u} \in U$$

Dynamical System (Mass & Energy Balance)

$$\frac{dC_m}{dt} = (1 + \epsilon C_m / C_{m_0}) R_m,$$
$$\frac{dC_i}{dt} = R_i + \epsilon C_i / C_{m_0} R_m,$$
$$\frac{dT}{dt} = \frac{\alpha_0 k_P \xi_0 C_m}{1 + \epsilon C_m / C_{m_0}} + \alpha_1 (T_j - T)$$

Relaxations of Dynamical System

Rate constants (R_m, R_i) from pseudo-empirical models

Rate Expression *(Greatly Simplified….)*

$$R_m = -C_m \xi_0 (k_P + k_{fm}),$$
$$R_i = -k_i C_i,$$

$$\xi_0^2 k_t(\xi_0, C_m, T) - 2\zeta k_i C_i = 0.$$

Use 3-layer GeLU ANN as in place of solving nonlinear equation from quasi-steady state assumption

Able to solve SIP in 57.8 s using a modified SIPres algorithm.

Conclusions

- EAGO is an extensible deterministic global optimization solver
  - Architected specifically for user-defined functions and routines
  - Performance comparable with state-of-the-art solvers
  - Open-source and free for non-commercial use
- Now and Near Future:
  - Exhaustive library of relaxation envelopes for commonly encountered subexpressions
  - Additional relaxations (αBB and AVM)
  - Release of dynamic optimization (optimal control) package
  - Integer variables
- Feature requests welcome on our GitHub!

Thank You – Any Questions?

- PSORLab@UCONN
- INFORMS 2021 Organizers
- Funding: National Science Foundation

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https://www.psor.uconn.edu

https://www.github.com/PSORLab/EAGO.jl

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Motivation: Reduced-Space Optimization

Want to solve dynamic optimization problems to guaranteed global optimality:

$$
\begin{aligned}
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\text{s.t.} \ & \dot{\mathbf{x}}(\mathbf{p}, t) = \mathbf{f}(\mathbf{x}(\mathbf{p}, t), \mathbf{p}, t), \forall t \in I = [t_0, t_f] \\
& \mathbf{x}(\mathbf{p}, t_0) = \mathbf{x}_0(\mathbf{p}) \\
& \mathbf{g}(\mathbf{x}(\mathbf{p}, t_f), \mathbf{p}) \leq \mathbf{0}
\end{aligned}
$$

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Parametric ordinary differential equation initial value problem (ODE-IVP) constraints.

Arise from optimal control, parameter estimation, etc.

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$$\dot{\mathbf{x}}(\mathbf{p},t) = \mathbf{f}(\mathbf{x}(\mathbf{p},t),\mathbf{p},t),\ \forall t \in I = [t_0, t_f]$$
$$\mathbf{x}(\mathbf{p},t_0) = \mathbf{x}_0(\mathbf{p})$$

$$\begin{aligned} \mathbf{z}_0 &= \mathbf{x}_0(\mathbf{p}) \\ \hat{\mathbf{z}}_1 - \mathbf{z}_0 - h\mathbf{f}(\hat{\mathbf{z}}_1,\mathbf{p},t_1) &= \mathbf{0} \\ &\vdots \\ \hat{\mathbf{z}}_K - \hat{\mathbf{z}}_{K-1} - h\mathbf{f}(\hat{\mathbf{z}}_K,\mathbf{p},t_K) &= \mathbf{0} \end{aligned}$$

Discrete-time reformulation (implicit Euler)

Pa

constraints.

Arise from optimal control, parameter estimation, etc.

Motivation: Reduced-Space Optimization

Want to solve dynamic optimization problems to guaranteed global optimality:

$$\begin{aligned}
\phi^* = &\min_{\mathbf{p}\in P\subset\mathbb{R}^{n_p}} \phi(\mathbf{x}(\mathbf{p},t_f),\mathbf{p})\\
\text{s.t. } &\dot{\mathbf{x}}(\mathbf{p},t) = \mathbf{f}(\mathbf{x}(\mathbf{p},t),\mathbf{p},t),\ \forall t\in I=[t_0,t_f]\\
&\mathbf{x}(\mathbf{p},t_0) = \mathbf{x}_0(\mathbf{p})\\
&\mathbf{g}(\mathbf{x}(\mathbf{p},t_f),\mathbf{p}) \le \mathbf{0}
\end{aligned}$$

Dimensionality: n_p

$$\begin{aligned}
\phi^* = &\min_{\mathbf{p}\in P,\hat{\mathbf{z}}\in Z} \phi(\hat{\mathbf{z}},\mathbf{p},t_f)\\
\text{s.t. } &\mathbf{z}_0 = \mathbf{x}_0(\mathbf{u},\mathbf{p})\\
&\hat{\mathbf{z}}_1 - \mathbf{z}_0 - h\mathbf{f}(\hat{\mathbf{z}}_1,\mathbf{p},t_1) = \mathbf{0}\\
&\vdots \qquad \vdots\\
&\hat{\mathbf{z}}_K - \hat{\mathbf{z}}_{K-1} - h\mathbf{f}(\hat{\mathbf{z}}_K,\mathbf{p},t_K) = \mathbf{0}\\
&\mathbf{g}(\hat{\mathbf{z}}_K,\mathbf{p}) \le \mathbf{0}
\end{aligned}$$

Dimensionality: $n_p \times K$

Motivation: Reduced-Space Optimization

Want to solve dynamic optimization problems to guaranteed global optimality:

$$
\begin{aligned}
\phi^* = \min_{\mathbf{p}\in P\subset\mathbb{R}^{n_p}} \ & \phi(\mathbf{x}(\mathbf{p},t_f),\mathbf{p}) \\
\text{s.t.} \ & \dot{\mathbf{x}}(\mathbf{p},t) = \mathbf{f}(\mathbf{x}(\mathbf{p},t),\mathbf{p},t),\ \forall t \in I = [t_0,t_f] \\
& \mathbf{x}(\mathbf{p},t_0) = \mathbf{x}_0(\mathbf{p}) \\
& \mathbf{g}(\mathbf{x}(\mathbf{p},t_f),\mathbf{p}) \le \mathbf{0}
\end{aligned}
$$

Dimensionality: n_p

$$
\begin{aligned}
\phi^* = \min_{\mathbf{p}\in P,\hat{\mathbf{z}}\in Z} \ & \phi(\hat{\mathbf{z}},\mathbf{p},t_f) \\
\text{s.t.} \ & \mathbf{z}_0 = \mathbf{x}_0(\mathbf{u},\mathbf{p}) \\
& \hat{\mathbf{z}}_1 - \mathbf{z}_0 - h\mathbf{f}(\hat{\mathbf{z}}_1,\mathbf{p},t_1) = \mathbf{0} \\
& \vdots \qquad \vdots \\
& \hat{\mathbf{z}}_K - \hat{\mathbf{z}}_{K-1} - h\mathbf{f}(\hat{\mathbf{z}}_K,\mathbf{p},t_K) = \mathbf{0} \\
& \mathbf{g}(\hat{\mathbf{z}}_K,\mathbf{p}) \le \mathbf{0}
\end{aligned}
$$

Dimensionality: $n_p \times K$

$$
\begin{aligned}
\phi^* = \min_{\mathbf{p}\in P} \ & \phi(\mathbf{z}(\mathbf{p}),\mathbf{p},t_f) \\
\text{s.t.} \ & \mathbf{g}(\mathbf{z}_K(\mathbf{p}),\mathbf{p}) \le \mathbf{0}
\end{aligned}
$$

$$\mathbf{z}(\mathbf{p}) = (\mathbf{z}_0(\mathbf{p}),\mathbf{z}_1(\mathbf{p}),\ldots,\mathbf{z}_K(\mathbf{p}))$$